582e: Lectures 4-5: Morphisms of Pairs.

§1. Morphisms of Pairs

$$(G, A) \longrightarrow (G', A')$$

: $G' \xrightarrow{f} G$, $A' \xleftarrow{g} A$ such that $g(f(s').a) = s'.g(a)$ for all $s' \in G'$, $a \in A$

induces:

$$H^q(G,A) \longrightarrow H^q(G',A') \quad \text{for all } q.$$

Why: $f : G' \to G$ induces a morphism of the resolutions of $\mathbb{Z}$ in Mod_G and $\mathrm{Mod}_{G'}$.

$$\begin{tikzcd}
\cdots \arrow[r] & \mathbb{Z}[G'\times G'] \arrow[r] \arrow[d] & \mathbb{Z}[G'] \arrow[r] \arrow[d, "f"] & \mathbb{Z} \arrow[r] & 0 \\
\cdots \arrow[r] & \mathbb{Z}[G\times G] \arrow[r] & \mathbb{Z}[G] \arrow[r] & \mathbb{Z} \arrow[r] & 0
\end{tikzcd}$$

Hence get morphism of corresponding complexes (in other direction).

$$\begin{tikzcd}
\mathrm{Hom}_G(\mathbb{Z}[G], A) \arrow[r] \arrow[d] & \mathrm{Hom}_G(\mathbb{Z}[G\times G], A) \arrow[r] \arrow[d] & \cdots \\
\mathrm{Hom}_{G'}(\mathbb{Z}[G'], A') \arrow[r] & \mathrm{Hom}_{G'}(\mathbb{Z}[G'\times G'], A') \arrow[r] & \cdots
\end{tikzcd}$$

$$(\varphi : \mathbb{Z}[G] \to A) \longmapsto (g \circ \varphi \circ f : \mathbb{Z}[G'] \longrightarrow A')$$

$$\begin{tikzcd}
\mathbb{Z}[G'] \arrow[rr, "\cancel{A}"] \arrow[dr, "f"'] & & A' \\
& \mathbb{Z}[G] \xrightarrow{\varphi} A \arrow[ur, "g"'] &
\end{tikzcd}$$

Morphism of pairs $(G,A) \longrightarrow (G',A')$ induces morphism on cohomology $H^q(G,A) \longrightarrow H^q(G',A')$.

Useful!!!

§2. Shapiro's Lemma:

Special case:

$G' \leq G$ subgroup

A' — G'-module.

$A = \mathrm{Hom}_{G'}(\mathbb{Z}[G], A')$ coinduction to make a left G-module from A'.

action: $\varphi : \mathbb{Z}[G] \to A'$

$s \in G$

then $s.\varphi : t \mapsto \varphi(ts^{-1})$

For fixed s, $\psi_s : \mathbb{Z}[G] \to \mathbb{Z}[G]$ given by $t \mapsto ts$ is a left G-module morphism,

$\varphi \mapsto \varphi \circ \psi_s$ would thus give a right G-action since it is precomp., so is a left action.

Note: Very important in making Eichler Shimura explicit and computable...

Shapiro's Lemma: For all $q \geq 0$,

$$H^q(G, A) \cong H^q(G', A').$$

Proof: $L^\bullet \twoheadrightarrow \mathbb{Z}$ free $\mathbb{Z}[G]$-resolution,

$\Rightarrow$ $L^\bullet \twoheadrightarrow \mathbb{Z}$ also free $\mathbb{Z}[G']$-resolution, since G' is a subgroup.

We have

$$\mathrm{Hom}_G(L^i, A) \cong \mathrm{Hom}_{G'}(L^i, A') \ni \varphi$$

$$\| $$

$$\mathrm{Hom}_G(L^i, \mathrm{Hom}_{G'}(\mathbb{Z}[G], A'))$$

$$b \mapsto (s \mapsto \varphi(bs^{-1}))$$

ψ: $b \mapsto \psi(b)(1) \in A'$.

$s' b \mapsto \psi(s'b)(1)$

$= s'\psi(b)(1)$

Conclusion: Same complexes so same cohomology.

§3: Inf and Res.

$H \hookrightarrow G$

$H \leq G$ subgroup

A – G-module

Morphism of pairs: $(G, A) \longrightarrow (H, A)$

$$H \hookrightarrow G \qquad A \xleftarrow{id} A$$

induces: $H^q(G, A) \xrightarrow{res_H} H^q(H, A)$.

"Restriction homomorphism".

E.g.: in terms of 1-cocycles

$$H^1(G,A) \xrightarrow{res_H} H^1(H,A)$$

$$[f] \longmapsto [f|_H] \quad \text{literally "restriction".}$$

$f : G \to A$

$f(st) = f(s) + s f(t)$

Next suppose $H \trianglelefteq G$. Get $(G/H, A^H) \longrightarrow (G, A)$

$$\begin{tikzcd} G \arrow[r] \arrow[d, dash] & G/H \arrow[d, dash] \\ A & A^H \arrow[l, hook'] \end{tikzcd}$$

Induces the inflation homomorphism:

$$H^q(G/H, A^H) \xrightarrow{inf} H^q(G,A).$$

Theorem (for later):

$$0 \to H^1(G/H, A^H) \xrightarrow{inf} H^1(G,A) \xrightarrow{res} H^1(H,A)^{G/H} \xrightarrow{Tr} H^2(G/H, A^H)$$

is exact.

Full Proof uses spectral sequences.

§4. Inner Automorphisms:

Suppose $t \in G$

$s \longmapsto tst^{-1}$ inner automorphism of G

Morphism of pairs: $(G,A) \longrightarrow (G,A)$

$$\begin{array}{ccc} tst^{-1} & \longleftarrow\!\shortmid & s \\ G & \longleftarrow & G \\ | & & | \\ a & \longmapsto & t^{-1}a \\ A & \longrightarrow & A \end{array}$$

Get $H^q(G,A) \longrightarrow H^q(G,A)$.

i.e. a natural action of G on $H^q(G,A)$!

$st^{-1}a$

$tst^{-1}a$

Prop: This action is trivial, i.e. map $t \in G$ induces identity map on $H^q(G,A)$.

Proof: Use dimension shifting.

$q=0$:

$$\begin{array}{ccc} H^0(G,A) \cong \mathrm{Hom}_G(\mathbb{Z}[G],A) \xrightarrow{d} & & \varphi: \mathbb{Z}[G] \to A \text{ with } \varphi \in \ker(d) \text{ so } \varphi(1) \in A^G \\ \downarrow & & \downarrow \\ H^0(G,A) \cong \mathrm{Hom}_G(\mathbb{Z}[G],A) \xrightarrow{d} & & g \circ \varphi \circ f \end{array}$$

where $f: G \to G$, $f(s) = tst^{-1}$

$g: A \to A$, $g(a) = t^{-1}(a)$

so $(g \circ \varphi \circ f)(1)$
$= g(\varphi(t 1 t^{-1}))$
$= t^{-1}\varphi(t 1 t^{-1}) = t^{-1}(\varphi(1))$
$= \varphi(1)$
since $\varphi(1)$ is G-invariant.

so map is id for $q=0$.

$q>0$: $0 \to A \longrightarrow A^* \longrightarrow A' \to 0$ with A^* co-induced.

$H^q(G,A) \cong H^{q-1}(G,A')$ for $q \geq 2$

and $H^1(G,A) \cong \mathrm{coker}(H^0(G,A^*) \to H^0(G,A'))$

|| functorial.

By induction, $H^{q-1}(G,A') \xrightarrow{\cong} H^{q-1}(G,A')$

by conj. by t, so prop follows.