582e Lecture 3: $H^q(G,A) = \dfrac{\text{cocycles}}{\text{coboundaries}}$

G arbitrary group.

Free resolution of $\mathbb{Z}$ as G-module:

$$(*) \qquad \cdots \longrightarrow P_2 \xrightarrow{d} P_1 \xrightarrow{d} P_0 \xrightarrow{d} \mathbb{Z} \to 0$$

$P_i = \mathbb{Z}[\underbrace{G\times\cdots\times G}_{i+1 \text{ copies}}]$ with $s.(g_0,\ldots,g_{i+1}) = (sg_0,\ldots,sg_{i+1})$

$d(g_0,\ldots,g_i) = \sum_{j=0}^{i}(-1)^j(g_0,\ldots,\hat{g}_j\cdots g_i)$.

Lemma: P_i is a free G-module

Proof: Basis: $\{(1,g_1,g_1g_2,\ldots,g_1\cdots g_i) : g_1,\ldots,g_i\in G\}$ is a basis over $\mathbb{Z}[G]$.

Prop: $(*)$ is exact.

dd = 0: standard calculation from defn's, e.g.

$$(dd)(g_0,g_1,g_2) = d\big((g_1,g_2) - (g_0,g_2) + (g_0,g_1)\big)$$
$$= \big((g_2)-(g_1)\big) - \big((g_2)-(g_0)\big) + (g_1) - (g_0) = 0.$$

(similar in general)

So $\operatorname{im}(d_{i+1}) \subseteq \operatorname{Ker}(d_i)$.

Fix $s\in G$.
Let $h: P_i \to P_{i+1}$ by $h(g_0,\ldots,g_i) = (s,g_0,\ldots,g_i)$.

Claim: $dh + hd = 1$.

Proof: $(dh+hd)(g_0,\ldots,g_i) = (dh)(g_0,\ldots,g_i) + hd(g_0,\ldots,g_i)$

$$= d(s,g_0,\ldots,g_i) + \sum_{j=0}^{i}(-1)^j(s,g_0,\ldots\hat{g}_j,\ldots,g_i)$$

$$= (g_0,\ldots,g_i) + \sum_{j=1}^{i+1}(-1)^{j+1}(s,g_0,\ldots\hat{g}_j,\ldots g_i) + \ldots = (g_0,\ldots g_i) \checkmark$$

So if $x\in \operatorname{Ker}(d_i)$ then

$$dhx + h\cancel{dx}^{\,0} = x \Rightarrow d(hx) = x \text{ so } x\in \operatorname{Im}(d_{i+1}). \quad \blacksquare$$

$H^q(G,A)$:

$\mathrm{Hom}_G(-,A)$ contravariant functor.

Let $K_i = \mathrm{Hom}_G(P_i, A)$

$$0 \to K_0 \xrightarrow{d_0} K_1 \xrightarrow{d_1} K_2 \xrightarrow{d_2} K_3 \xrightarrow{d_3} \cdots$$ complex

$$\boxed{H^q(G,A) = \ker(d_q)/\mathrm{im}(d_{q-1})}$$

$f \in \mathrm{Hom}_G(P_i, A)$. P_i free on $(1, g_1, g_1g_2, \ldots, g_1\cdots g_i)$

Let $\varphi(g_1, \ldots, g_i) = f(1, g_1, g_1 g_2, \ldots, g_1 \cdots g_i)$, where $\varphi : G^i \to A$ set-theoretic map.

Prop: $(d\varphi)(g_1, \ldots g_{i+1}) = g_1 \varphi(g_2, g_3, \ldots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j \varphi(g_1, g_2, \ldots, g_j g_{j+1}, \ldots, g_{i+1})$ (prod of g_j and g_{j+1})

$$+ (-1)^{i+1} \varphi(g_1, \ldots, g_i).$$

$f : P_i \to A$

Proof: $\varphi : P_i \to A$

$$\varphi \circ d \longleftarrow \varphi$$

$P_{i+1} \longrightarrow P_i$ so $\mathrm{Hom}(P_{i+1}, A) \longleftarrow \mathrm{Hom}(P_i, A)$

$$(d\varphi)(g_1, \ldots, g_{i+1}) = f(d(1, g_1, g_1g_2, \ldots, g_1 \cdots g_{i+1}))$$

$$= f\Big((g_1, g_1g_2, \ldots, g_1 \cdots g_{i+1}) + \sum_{j=1}^{i} (-1)^j (1, g_1, \ldots, \widehat{g_1 \cdots g_j}, g_1 \cdots g_{j+1}, \ldots, g_1 \cdots g_{i+1})$$

$$+ (-1)^{i+1} \cdot (1, g_1, g_1 g_2, \ldots, g_1 \cdots g_i)\Big)$$

$$= g_1 \varphi(g_2, g_3, \ldots, g_{i+1}) + \sum_{j=1}^{i} (-1)^j \varphi(g_1, \ldots, g_j g_{j+1}, \ldots, g_{i+1})$$

$$+ (-1)^{i+1} \varphi(g_1, \ldots, g_i)$$

Explicit:

Case $i=0$:

$\varphi \in \mathrm{Hom}(P_0, A)$, so there's $a \in A$ s.t. $\varphi() = a$. (function of 0 variables.)

$$(d\varphi)(g) = g\varphi() - \varphi() = ga - a.$$

Case $i=1$:

$\varphi \in \mathrm{Hom}(P_1, A)$.

$$(d\varphi)(g_1, g_2) = g_1\varphi(g_2) - \varphi(g_1 g_2) + \varphi(g_1).$$

Thus:

$$H^0(G,A) = A^G$$

$$H^1(G,A) = \{\text{1-cocycles}\}/\{\text{1-coboundaries}\}$$

$$= \{\varphi : G \to A \;:\; d\varphi = 0\} \big/ \{d\varphi \;:\; \varphi \in \mathrm{Hom}(P_0, A)\}$$

$$= \frac{\{\varphi : G \to A \;:\; \varphi(g_1 g_2) = \varphi(g_1) + g_1\varphi(g_2)\}}{\{\varphi_a : G \to A \text{ all } a \in A\}}$$

where $\varphi_a(g) = ga - a$.

$$H^2(G,A) = \frac{\{\varphi : G \times G \to A \;:\; g_1 \cdot \varphi(g_2, g_3) - \varphi(g_1 g_2, g_3) + \varphi(g_1, g_2 g_3) - \varphi(g_1, g_2) = 0\}}{\cdots}$$

such φ arise by taking set-theoretic section σ

$$1 \to A \to E \xrightarrow{\sigma\ \text{(section)}} G \to 1$$

$$\varphi(g_1, g_2) = \frac{\sigma(g_1)\sigma(g_2)}{\sigma(g_1 g_2)}.$$

$G' \to G$
$A' \leftarrow A$
$H^q(G,A) \to H^q(G', A')$

Next: Either "explicit computation of H^1" or "Morphisms of pairs". No class next Fri.