Math 414 Lecture 5

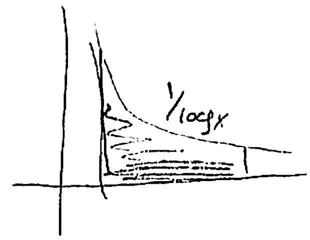
§1 Recall:

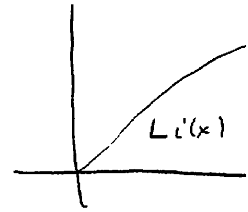
Prime numbers: $\{2,3,5,\dots\}$

$\pi(x) = \#\{p \text{ prime} : p \le x\}$

stairstep function

$Li(x) = \int_2^x \frac{1}{\log(x)}\,dx$





Conjecture (The Riemann Hypothesis):

For all $x \ge 2.01$,

$$\left|\pi(x) - Li(x)\right| \le \sqrt{x}\cdot\log(x)$$

RH

Unsolved
- No proof in view
- But lots of "cranks".

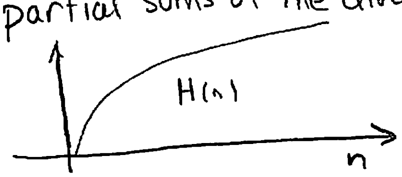
§2. An elementary reformulation

Point: Show how RH can pop up in surprising ways.

Consider: Let $H_n = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}$ (the n-th "harmonic number")

(partial sums of the divergent series $\sum \frac{1}{n}$)

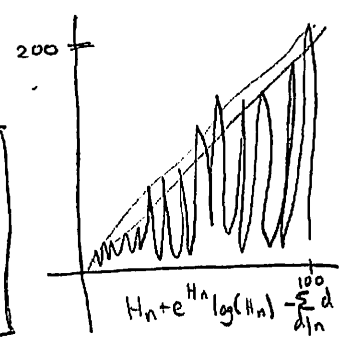
n	H_n
1	1
2	3/2
10	7381/2520



Conjecture: For each $n > 1$,

$$\sum_{d \mid n} d < H_n + e^{H_n}\cdot\log(H_n).$$

positive divisors



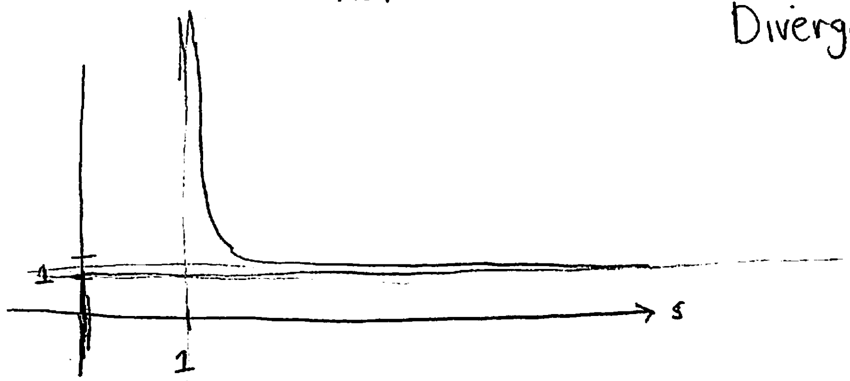
Theorem (Lagarias, 2001): Above conj. is equivalent to RH.

§ 3. The Riemann Zeta function.

Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$.

Converges for $s > 1$.

Diverges at $s = 1$. (Harmonic series again)



A very short course in complex analysis:

$D \subseteq \mathbb{C}$ (nice) subset

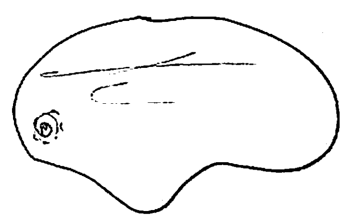
Defn: $f : D \longrightarrow \mathbb{C}$ is analytic if for every $z_0 \in D$ we have

$$f(z) = \sum_{n \geq 0} a_n (z - z_0)^n \qquad \text{fixed } a_n \in \mathbb{C}.$$

in some disc around z_0, I.e. f is (locally) a convergent power series

Fact: D : nice connected

Then: If $f \equiv 0$ on a little disk then $f \equiv 0$ on all D.



Theorem: There is a unique analytic function

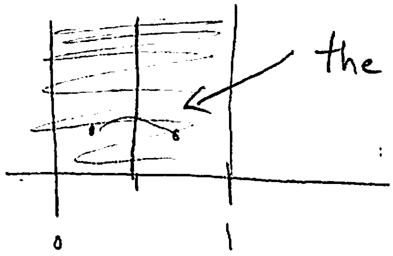
$$\zeta(s) : \mathbb{C} \setminus \{1\} \longrightarrow \mathbb{C}$$

that agrees with $\sum \frac{1}{n^s}$ for s with $\mathrm{Re}(s) > 1$.

Functional Equation:

Let $\xi(s) = \pi^{-s/2}\,\Gamma\left(\frac{s}{2}\right)\zeta(s)$.

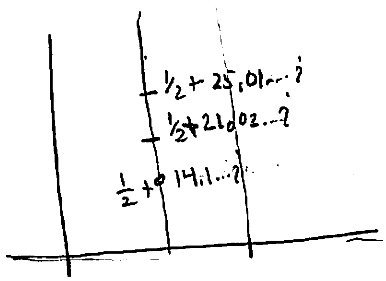
Then $\xi(s) = \xi(1-s)$.



the critical strip

Conjecture (Riemann Hypothesis):

The zeros z of $\zeta(s)$ with $Re(z) > 0$ all lie on the line $Re(s) = \frac{1}{2}$.



i.e. If $\zeta(z) = 0$ and $Re(z) > 0$ then $Re(z) = \frac{1}{2}$.

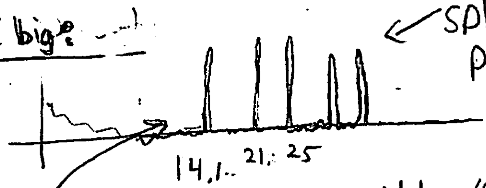
§4. Spectrum of the Prime Numbers

C some constant

Let $F_C(x) = -\sum_{p^n < C} p^{-n/2} \log(p) \cos(n \cdot \log(p)\, x)$ ← A classi… Fourier… got by Fourier transfor… of a f… very si… to $\pi(x$…

prime power

Plot for C big:



← spikes at imaginary parts of zeros above.

…iemann: These numbers are the hidden "spectrum" of the primes.

(Computer demo if time permits.)