MATH 414: Lecture 4: What is Riemann's Hypothesis?

Primes: $p_1 = 2$, $p_2 = 3$, $p_3 = 5$, $p_4 = 7$, $p_5 = 11$, ..., p_n = n-th prime.

Theorems:
- Every integer factors uniquely as a product of primes
- There are infinitely many primes.

Some Questions About Primes: (also use computer?)

- **Twin Primes:** ~~unsolved~~

  Open problem: Are there infinitely many twin primes

  p_n, $p_n + 2$

  both prime? (Conj: yes)

  Ex: 3,5 5,7 11,13 17,19 29,31, ...

n	primes p_n s.t. $p_n + 2$ also prime
100	8
10^3	35
10^4	205
10^5	1224
10^6	8169
10^7	58980

- **Goldbach's Conjecture:**

  Open Problem: Every even integer ≥ 4 is a sum of 2 primes.

  E.g. $4 = 2+2$, $6 = 3+3$, $8 = 3+5$, $10 = 5+5$, $12 = 5+7$

  $2010 = \underline{7} + \underline{2003}$

Mersenne Primes:

Prime of the form 2^p-1 with p prime.

$2^2-1=3$

$2^3-1=7$

$2^5-1=31$ Only 47 are known!!

$2^7-1=127$

$2^{11}-1=2047=23\cdot 89$

Open Problem: Infinitely many Mersenne primes?

Prime Gaps:

$\mathrm{Gap}_k(X) = \#\{\text{primes } p_n \le X \text{ s.t. } p_n - p_{n-1} = k\}$

(so, $\mathrm{Gap}_2(X) = \#\{\text{twin prime pairs} \le X\}$)

X	$\mathrm{Gap}_2(X)$	$\mathrm{Gap}_4(X)$	$\mathrm{Gap}_6(X)$	$\mathrm{Gap}_8(X)$
10^2	8	7	7	1
10^4	205	202	299	101
10^6	8169	8143	13549	5569

Which gap size is most popular?

Conj (Hardy-Littlewood) Eventually, $30=2\cdot3\cdot5$ will be most pop. Then $210=2\cdot3\cdot5\cdot7$ will take over... then 2310, . . etc.

These are all questions about how primes are distributed. They have stumped humanity for aeons.

Recall:

$\pi(x) = \#\{p \text{ prime} : p \leq x\}$

x	$\pi(x)$
10	4
10^2	25
10^3	168
10^4	1229
10^5	9592
10^6	78498

<u>Theorem (Prime Number Theorem)</u>: 1896 <u>Hadamard</u> & <u>de la Vallée Poussin</u>

$$\pi(x) \sim \frac{x}{\log(x)-1}$$

i.e. $\lim_{x\to\infty} \pi(x) / (x/(\log(x)-1)) = 1$.

(see Wikipedia page for link to short proof in paper of Zagier)

possible project if you know complex analysis.

<u>Ex</u>:

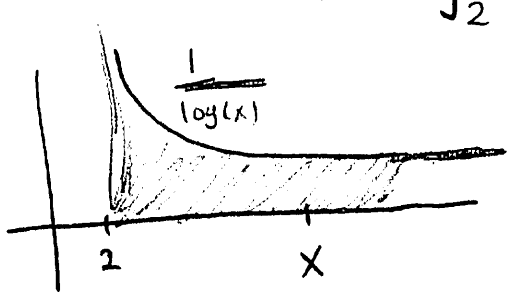
x	$\sim \pi(x)/(x/(\log(x)-1))$
10	0.52
10^3	0.995
10^6	1.006
10^9	1.00287
10^{10}	1.00229
10^{11}	1.001858

$\pi(10^{11}) = 4118054813$

$10^{11}/(\log(10^{11})-1) = 4110416300.732\ldots$

Gauss had a better guess at a function to approx $\pi(x)$:

$$Li(x) = \int_2^x \frac{dx}{\log(x)}$$



<u>Gauss's Guess</u>

$Li(x) \sim \pi(x)$ better.

Suppose we want to approximate something.

"Good" approximation $g(x)$ to $f(x)$: "square root error", meaning "about half digits right".

Precisely: $$|g(x) - f(x)| \leq \sqrt{x} \cdot \log(x)$$

$g(x)$ is a good approx to $f(x)$.

Observe: Maybe $\frac{x}{\log(x)-1}$ is a good approx to $\pi(x)$?

$x = 10^{11}$:

$$\left|\pi(x) - \frac{x}{\log(x)-1}\right| \doteq 7{,}638{,}512\ldots$$

just slips by.

$$\sqrt{x} \cdot \log(x) : \doteq 8{,}009{,}554$$

$x = 10^{12}$: $$|\pi(x) - x/\log(x)-1| \doteq 57718368.0\ldots$$

Oops.

$$\sqrt{x} \cdot \log(x) \doteq 27631021.115\ldots$$

Try Li(x) instead:

$x = 10^{12}$: $$|\pi(x) - \mathrm{Li}(x)| \doteq 38187.48\ldots$$

Woot!

room to spare.

Conjecture (Riemann's Hypothesis):

For all $x \geq 2.01$,

$$|\pi(x) - \mathrm{Li}(x)| \leq \sqrt{x} \cdot \log(x)$$