Exercise Set 7:

Continued Fractions

Math 414, Winter 2010, University of Washington

Due Wednesday, February 24, 2010

1. Show that every nonzero rational number can be represented in exactly two ways by a finite simple continued fraction. (For example, 2 can be represented by $[1,1]$ and $[2]$, and $1/3$ by $[0,3]$ and $[0,2,1]$.)

2. If $c_n = p_n/q_n$ is the nth convergent of $[a_0, a_1, \ldots, a_n]$ and $a_0 > 0$, show that
$$[a_n, a_{n-1}, \ldots, a_1, a_0] = \frac{p_n}{p_{n-1}}$$
and
$$[a_n, a_{n-1}, \ldots, a_2, a_1] = \frac{q_n}{q_{n-1}}.$$
(Hint: In the first case, notice that $\frac{p_n}{p_{n-1}} = a_n + \frac{p_{n-2}}{p_{n-1}} = a_n + \frac{1}{\frac{p_{n-1}}{p_{n-2}}}$.)

3. Use continued fractions to find an "impressive" rational approximation to $e^{\pi} = 23.1406\ldots$.

4. Evaluate the infinite continued fraction $[2, \overline{1,2,1}]$.

5. Determine the infinite continued fraction of $\frac{1+\sqrt{13}}{2}$.

6. Let d be an integer that is coprime to 10. Prove that the decimal expansion of $\frac{1}{d}$ has a period equal to the order of 10 modulo d. (Hint: For every positive integer r, we have $\frac{1}{10^r-1} = \sum_{n\geq 1} 10^{-rn}$.)

7. Find a positive integer that has at least three different representations as the sum of two squares, disregarding signs and the order of the summands.