Math 480 (Spring 2007): Homework 2

Due: Monday, April 9

There are 8 problems. Each problem is worth 6 points and parts of multipart problems are worth equal amounts. (Note: 6 points each, instead of 5 this time.)
Office Hours. My official office hours are on Thursdays 4–6pm in Padelford C423.

1. (a) Prove that for any positive integer n, the set $(\mathbb{Z}/n\mathbb{Z})^*$ under multiplication modulo n is a group.

   (b) Prove that for any positive integer n, the set $\mathbb{Z}/n\mathbb{Z}$ under addition and multiplication modulo n is a ring.

2. Prove that for every positive integer n the integer $5^{2n} + 3 \cdot 2^{5n-2}$ is divisible by 7.

3. Let $f(x) = x^3 + a \in \mathbb{Z}[x]$ be a cubic polynomial with integer coefficients, e.g., $f(x) = x^3 + 1$.

   (a) Formulate a conjecture about when the set $\{f(n) : n \in \mathbb{Z} \text{ and } f(n) \text{ is prime}\}$ is infinite.

   (b) Give numerical evidence that supports your conjecture.

4. Prove the following statements:

   (a) If a is an odd integer, then $a^2 \equiv 1 \pmod{8}$.

   (b) For any integer a, we have $a^3 \equiv 0, 1, \text{ or } 6 \pmod{7}$.

   (c) For any integer a, we have $a^4 \equiv 0 \text{ or } 1 \pmod{5}$.

5. Find rules for divisibility of an integer by 5, 9, and 11, and prove each of these rules using arithmetic modulo a suitable n.

6. What is the multiplicative order of 3 modulo 17? (You may use any method (even a computer) to answer this question, as long as you explain what you do.)

7. A basket has n eggs in it. One egg remains when the eggs are removed from the basket 2, 3, 4, 5, or 6 at a time. No egg remains if they are removed 7 at a time. Find the smallest number n of eggs that could be in the basket.

8. (a) Verify by hand the equation $\sum_{d|n} \varphi(d) = n$ in the case $n = 12$. Here we are summing over the positive divisors of n.

   (b) Verify by computer that $\sum_{d|n} \varphi(d) = n$ for $n = 1000$. In your solution make sure to explain exactly what you type into the computer, and what program you use.