Computing Heegner Points I

<u>Ref</u>: • §8.6 of Cohen's new book, Number theory I, II
by Delaunay
• which is almost identical to Watkin's paper

E - elliptic curve over $\mathbb{Q}$

$f_E = \sum_{n=1}^{\infty} a_n q^n$ modular form

$$\begin{tikzcd} \mathfrak{h} \arrow[d, "{\tilde{\Phi}(\tau) = 2\pi i \int_{i\infty}^{\tau} f_E(z)dz \;=\; \sum_{n\geq 1} \frac{a_n}{n} q^n}"] \\ \mathbb{C}/\Lambda \arrow[d, "{(\wp, \wp')}"] \\ E(\mathbb{C}) \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{h} \arrow[r] \arrow[dr, "\tilde{\Phi}"'] & X_0(N)_{\mathbb{C}} = \mathfrak{h}^* / \Gamma_0(N) \arrow[d, "\phi"] \\ & \mathbb{C}/\Lambda \end{tikzcd}$$

<u>Heegner Point</u>:

K - q.i. field

$\mathcal{N} \subseteq \mathcal{O}_K$ ideal with $\mathcal{O}_K/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$

<u>Heegner Point</u>: $\left[\left(\mathbb{C}/\mathcal{O}_K, \mathcal{N}^{-1}/\mathcal{O}_K\right)\right] \in X_0(N)(\mathbb{C}) = \Gamma_0(N)\backslash\mathfrak{h}$.

Alternative concrete perspective (starting with Birch)

<u>Defn</u>: • $\tau \in \mathfrak{h}$ is a <u>CM point</u> if τ is a root of an equation

$$Ax^2 + Bx + C = 0 \text{ with } A, B, C \in \mathbb{Z}, \quad B^2 - 4AC < 0$$

- If so, choose A, B, C so $\gcd(A,B,C) = 1$ and $A > 0$ – so unique –
  The <u>discriminant</u> of τ is $\Delta(\tau) = B^2 - 4AC$.

• $\tau \in \mathfrak{h}$ is a <u>Heegner point</u> of level N if $\Delta(N\tau) = \Delta(\tau)$.

Prop: (1) If $\gamma \in SL_2(\mathbb{Z})$ then
$$\Delta(\gamma(\tau)) = \Delta(\tau)$$
for all $\tau \in \mathfrak{h}$

easy to see using determinants, since $\gamma(\tau)$ corr. to conj a matrix by γ, and disc is a det.

(2) If $\tau \in \mathfrak{h}$ is a Heegner point, so is $\gamma(\tau)$ for all $\gamma \in \Gamma_0(N)$.

Proof of 2: We have $\Gamma_0(N) = \Gamma \cap \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \Gamma \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ for $\Gamma = SL_2(\mathbb{Z})$

$$N\tau = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}(\tau)$$

Suppose $\gamma \in \Gamma_0(N)$ and write $\gamma = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}^{-1} \gamma' \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$ for $\delta \in SL_2(\mathbb{Z})$

Then $\Delta(\gamma'(N\tau)) = \Delta(N\tau)$ by (1).

$\Delta(N\tau) = \Delta(\tau)$ since τ is Heegner point

But $\gamma'\begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}\gamma$, so

$$\Delta(\gamma'(N\tau)) = \Delta(N(\gamma\tau)) = \Delta(\gamma\tau)$$

$\Delta(\tau) = \Delta(\gamma\tau)$ ∎

Prop: Let $\tau \in \mathfrak{h}$ be a quadratic irrational, with corresponding quadratic form (A,B,C) of discriminant $D = \Delta(\tau)$.

(τ is a Heegner point of level N)

$\Updownarrow$

$N \mid A$ and any of the following equivalent statements holds:

- $\gcd(\frac{A}{N}, B, CN) = 1$
- $\gcd(N, B, AC/N) = 1$
- $\exists F \in \mathbb{Z}$ s.t. $B^2 - 4NF = D$ with $\gcd(N, B, F) = 1$.

Assume D is a fundamental discriminant, i.e. disc of a maximal order of quad. imag. field, so $D \equiv 0$ or $1 \pmod 4$
and either • $4 \| D$, $\frac{D}{4} \equiv 2$ or $3 \pmod 4$, and $\frac{D}{4}$ is square free
or • D is square free.

Gauss: $K = \mathbb{Q}(\sqrt{D})$

$$\underbrace{Cl(K)}_{\frac{\text{(fractional ideals)}}{\text{principal fractional ideals}}} \xleftrightarrow{\cong} \left\{ \begin{array}{l} \text{primitive quad forms } (A,B,C) \\ \text{of discriminant } D \end{array} \right\} \Big/ SL_2(\mathbb{Z})\text{-equiv.}$$

$$\Big[\underbrace{\mathbb{Z} + \frac{-B+\sqrt{D}}{2A}\mathbb{Z}}_{\text{fractional ideal}}\Big] \longleftrightarrow \big[(A,B,C)\big].$$

Prop:

$$\left\{ \begin{array}{l} \text{Heegner Points of} \\ \text{Discriminant } D \text{ and} \\ \text{Level } N \end{array} \right\} \Big/ \Gamma_0(N) \xleftrightarrow{\cong} \left\{ (\beta, [\mathfrak{a}]) : \begin{array}{l} \beta \in \mathbb{Z}/2N\mathbb{Z}, \\ [\mathfrak{a}] \in Cl(K) \end{array} \right\}$$

such that
$b^2 \equiv D \pmod{4N}$
for all lifts b of β.

$$(\beta, [\mathfrak{a}])$$
$$\updownarrow$$

$$\tau = \frac{-B+\sqrt{D}}{2A} \longleftarrow\!\!\!\mid \quad (A,B,C) \text{ with } B \equiv \beta \pmod{2N} \text{ and } N \mid A$$

$$\tau \leftrightarrow (A,B,C) \longmapsto \begin{array}{l} \beta \equiv B \pmod{2N} \\ \mathfrak{a} = \mathbb{Z} + \mathbb{Z}\tau. \end{array}$$

Recall: $K = \mathbb{Q}(\sqrt{D})$

H = Hilbert class field of K

= maximal unramified abelian extension of K.

$\mathrm{Gal}(H/K) \cong \mathrm{Cl}(K)$

Heegner points in $\boxed{X_0(N)(H).}$

Theorem:

$$\mathrm{Cl}(K) \longrightarrow \mathrm{Gal}(H/K)$$
$$\mathfrak{b} \longmapsto \mathrm{Artin}(\mathfrak{b})$$

$\tau \Leftrightarrow (\beta, [\mathfrak{a}])$ Heegner point of level N and discriminant D

$$\varphi((\beta, [\mathfrak{a}]))^{\mathrm{Artin}(\mathfrak{b})} = \varphi((\beta, [\mathfrak{a}\mathfrak{b}^{-1}]))$$

Thus

$$y_K = \mathrm{Tr}_{H/K}(\varphi((\beta, [\mathfrak{a}]))) = \sum_{[\mathfrak{b}] \in \mathrm{Cl}(K)} \varphi((\beta, [\mathfrak{a}\mathfrak{b}^{-1}]))$$

$$= \sum_{[\mathfrak{b}] \in \mathrm{Cl}(K)} \varphi((\beta, [\mathfrak{b}])).$$

Lemma: In fact if $\varepsilon_E = -1$, then $y_K \in E(\mathbb{Q})$, since y_K is fixed by conjugation.

In book: ~~a~~ reference to reduction, enumeration, etc.