3.6 Computing a basis for $S_2(\Gamma_0(N))$

This section is about a method for using modular symbols to compute a basis for $S_2(\Gamma_0(N))$. It is not the most efficient for certain applications, but it is easy to explain and understand. See Section 3.7 for a method that takes advantage of deeper structure of $S_2(\Gamma_0(N))$.

Let $\mathbb{M}_2(\Gamma_0(N);\mathbb{Q})$ and $\mathbb{S}_2(\Gamma_0(N);\mathbb{Q})$ denote modular symbols and cuspidal modular symbols over $\mathbb{Q}$. Before we begin, we describe a simple but crucial fact about the relation between cusp forms and the Hecke algebra.

If $f = \sum b_n q^n \in \mathbb{C}[[q]]$ is a power series, let $a_n(f) = b_n$ be the n coefficient of f. Notice that a_n is a $\mathbb{C}$-linear map $\mathbb{C}[[q]] \to \mathbb{C}$.

As explained in [Lan95, §VII.3] (recall also 2.4.6), the Hecke operators T_n acts on elements of $M_2(\Gamma_0(N))$ as follows:

$$T_n\left(\sum_{m=0}^{\infty} a_m q^m\right) = \left(\sum_{1\leq d \,|\, \gcd(n,m)} \varepsilon(d)\cdot d\cdot a_{mn/d^2}\right) q^m, \tag{3.6.1}$$

where $\varepsilon(d) = 1$ if $\gcd(d,N) = 1$ and $\varepsilon(d) = 0$ if $\gcd(d,N) \neq 1$. (Note: More generally, if $f \in M_2(\Gamma_1(N))$ is a modular form with Dirichlet character ε, then the above formula holds; above we are considering this formula in the special case when ε is the trivial character.)

Lemma 3.6.1. *Suppose f is a modular form and n is a positive integer. Then*

$$a_1(T_n(f)) = a_n(f).$$

Proof. The coefficient of q in (3.6.1) is $\varepsilon(1)\cdot 1\cdot a_{1\cdot n/1^2} = a_n$. □

Let $\mathbb{T}'$ denote the image of the Hecke algebra in $\mathrm{End}(S_2(\Gamma_0(N)))$, and let $\mathbb{T}'_{\mathbb{C}} = \mathbb{T}' \otimes_{\mathbb{Z}} \mathbb{C}$ be the $\mathbb{C}$-span of the Hecke operators.

Proposition 3.6.2. *There is a perfect bilinear pairing of complex vector spaces*

$$S_2(\Gamma_0(N)) \times \mathbb{T}'_{\mathbb{C}} \to \mathbb{C}$$

given by

$$\langle f, t\rangle = a_1(t(f)).$$

Proof. The pairing is bilinear since both t and a_1 are linear. Suppose $f \in S_2(\Gamma_0(N))$ is such that $\langle f, t\rangle = 0$ for all $t \in \mathbb{T}'_{\mathbb{C}}$. Then in particular $\langle f, T_n\rangle = 0$ for each positive integer n. But by Lemma 3.6.1 we have

$$a_n(f) = a_1(T_n(f)) = 0$$

for all n; thus $f = 0$.

Next suppose that $t \in \mathbb{T}'_{\mathbb{C}}$ is such that $\langle f, t\rangle = 0$ for all $f \in S_2(\Gamma_0(N))$. Then $a_1(t(f)) = 0$ for all f. For any n, the image $T_n(f)$ is also a cuspform, so

$a_1(t(T_n(f))) = 0$ for all n and f. Finally $\mathbb{T}'$ is commutative and Lemma 3.6.1 together imply that for all n and f,

$$0 = a_1(t(T_n(f))) = a_1(T_n(t(f))) = a_n(t(f)),$$

so $t(f) = 0$ for all f. Thus t is the 0 operator. $\square$

By Proposition 3.6.2 there is an isomorphism of vector spaces

$$\Psi : S_2(\Gamma_0(N)) \xrightarrow{\cong} \mathrm{Hom}(\mathbb{T}'_{\mathbb{C}}, \mathbb{C}) \tag{3.6.2}$$

that sends $f \in S_2(\Gamma_0(N))$ to the homomorphism

$$t \mapsto a_1(t(f)).$$

For any linear map $\varphi : \mathbb{T}'_{\mathbb{C}} \to \mathbb{C}$, let

$$f_\varphi = \sum_{n=1}^{\infty} \varphi(T_n) q^n \in \mathbb{C}[[q]].$$

By Lemma 3.6.1, we have

$$\langle f_\varphi, T_n \rangle = a_1(T_n(f_\varphi)) = a_n(f_\varphi) = \varphi(T_n).$$

Thus f_φ is the q-expansion of the modular form $\Psi^{-1}(\varphi)$ that corresponds via (3.6.2) to φ. Conclusion: The cuspforms f_φ for φ running through a basis of $\mathrm{Hom}(\mathbb{T}'_{\mathbb{C}}, \mathbb{C})$, form a basis for $S_2(\Gamma_0(N))$.

> *We can compute $S_2(\Gamma_0(N))$ by computing $\mathrm{Hom}(\mathbb{T}', \mathbb{C})$, where we compute $\mathbb{T}'$ in any way we want, e.g., using a space that contains an isomorphic copy of $S_2(\Gamma_0(N))$.*

Algorithm 3.6.3 (Basis of Cuspforms)**.** *Given positive integers N and B, this algorithm computes a basis for $S_2(\Gamma_0(N))$ to precision $O(q^B)$.*

1. Compute the modular symbols space $\mathbb{M}_2(\Gamma_0(N); \mathbb{Q})$ via the presentation of Section 3.3.2.
2. Compute the subspace $\mathbb{S}_2(\Gamma_0(N); \mathbb{Q})$ of cuspidal modular symbols as in Section 3.5.
3. Let $d = \frac{1}{2} \cdot \dim \mathbb{S}_2(\Gamma_0(N); \mathbb{Q})$. This is the dimension of $S_2(\Gamma_0(N))$.
4. Let $[T_n]$ denote the matrix of T_n acting on some fixed basis of $\mathbb{S}_2(\Gamma_0(N); \mathbb{Q})$. For a matrix A, let $a_{ij}(A)$ denote the ij-th entry of A. For various integers i, j with $0 \le i, j \le d-1$, compute formal q-expansions

$$f_{ij}(q) = \sum_{n=1}^{B-1} a_{ij}([T_n]) q^n + O(q^B) \in \mathbb{Q}[[q]]$$

   until we find enough to span a space of dimension d (or exhaust all of them). These f_{ij} then form a basis for $S_2(\Gamma_0(N))$ to precision $O(q^B)$.

3.6.1 Examples

In this section we use SAGE to demonstrate Algorithm 3.6.3 for computing $S_2(\Gamma_0(N))$ for various N.

Example 3.6.4. The smallest N with $S_2(\Gamma_0(N)) \neq 0$ is $N = 11$.

```
sage: M = ModularSymbols(11); M.basis()
((1,0), (1,8), (1,9))
sage: S = M.cuspidal_submodule(); S
Dimension 2 subspace of a modular symbols space of level 11
```

We compute a few Hecke operators, then read off a nonzero cuspform, which forms a basis for $S_2(\Gamma_0(11))$:

```
sage: S.T(2).matrix()
[-2  0]
[ 0 -2]
sage: S.T(3).matrix()
[-1  0]
[ 0 -1]
```

Thus

$$f_{0,0} = q - 2q^2 - q^3 + \cdots \in S_2(\Gamma_0(11))$$

forms a basis for $S_2(\Gamma_0(11))$.

Example 3.6.5. We compute a basis for $S_2(\Gamma_0(33))$ to precision $O(q^6)$.

```
sage: M = ModularSymbols(33)
sage: S = M.cuspidal_submodule(); S
Dimension 6 subspace of a modular symbols space of level 33
```

Thus $\dim S_2(\Gamma_0(33)) = 3$.

```
sage: R.<q> = PowerSeriesRing(QQ)   # make q indeterminate of Q[[q]]
sage: f00 = sum(S.T(n).matrix()[0,0]*q^n for n in range(1,6)) + O(q^6)
sage: f00
q - q^2 - q^3 + q^4 + O(q^6)
```

This gives us one basis element of $S_2(\Gamma_0(33))$. It remains to find two others. We find

```
sage: f01 = sum(S.T(n).matrix()[0,1]*q^n for n in range(1,6)) + O(q^6)
sage: f01
-2*q^3 + O(q^6)
```

and

```
sage: f10 = sum(S.T(n).matrix()[1,0]*q^n for n in range(1,6)) + O(q^6)
sage: f10
q^3 + O(q^6)
```

This third one is (to our precision) a scalar multiple of the second, so we look further.

```
sage: f11 = sum(S.T(n).matrix()[1,1]*q^n for n in range(1,6)) + O(q^6)
sage: f11
q - 2*q^2 + 2*q^4 + q^5 + O(q^6)
```

This latter form is clearly not in the span of the first two. Thus we have the following basis for $S_2(\Gamma_0(33))$ (to precision $O(q^6)$):

$$\begin{aligned} f_{00} &= q - q^2 - q^3 + q^4 + \cdots \\ f_{11} &= q - 2q^2 + 2q^4 + q^5 + \cdots \\ f_{10} &= q^3 + \cdots \end{aligned}$$

Example 3.6.6. Next consider $N = 23$, where we have $d = \dim S_2(\Gamma_0(23)) = 2$. The command `q_expansion_cuspforms` computes matrices T_n and returns a function f such that $f(i,j)$ is the q-expansion of $f_{i,j}$ to some precision. (Note: for efficiency reasons, $f(i,j)$ in SAGE actually computes matrices of T_n acting on a basis for the linear dual of $\mathbb{S}_2(\Gamma_0(N))$.)

```
sage: M = ModularSymbols(23)
sage: S = M.cuspidal_submodule()
sage: S
Dimension 4 subspace of a modular symbols space of level 23
sage: f = S.q_expansion_cuspforms(6)
sage: f(0,0)
q - 2/3*q^2 + 1/3*q^3 - 1/3*q^4 - 4/3*q^5 + O(q^6)
sage: f(0,1)
O(q^6)
sage: f(1,0)
-1/3*q^2 + 2/3*q^3 + 1/3*q^4 - 2/3*q^5 + O(q^6)
```

Thus a basis for $S_2(\Gamma_0(23))$ is

$$\begin{aligned} f_{0,0} &= q - \frac{2}{3}q^2 + \frac{1}{3}q^3 - \frac{1}{3}q^4 - \frac{4}{3}q^5 + \cdots \\ f_{1,0} &= -\frac{1}{3}q^2 + \frac{2}{3}q^3 + \frac{1}{3}q^4 - \frac{2}{3}q^5 + \cdots \end{aligned}$$

Or, in echelon form,

$$\begin{aligned} &q - q^3 - q^4 + \cdots \\ &\quad q^2 - 2q^3 - q^4 + 2q^5 + \cdots \end{aligned}$$

which we computed using

```
sage: S.q_expansion_basis(6)
 [q - q^3 - q^4 + O(q^6),
    q^2 - 2*q^3 - q^4 + 2*q^5 + O(q^6)]
```

3.7 Computing $S_2(\Gamma_0(N))$ using eigenvectors

In this section we describe how to use modular symbols to construct a basis of $S_2(\Gamma_0(N))$ consisting of modular forms that are eigenvectors for every element of the ring $\mathbb{T}'$ generated by the Hecke operator T_p, with $p \nmid N$. Such eigenvectors are called *eigenforms*.

Suppose M is a positive integer that divides N. As explained in [Lan95, VIII.1–2], for each divisor d of N/M there is a natural *degeneracy map* $\beta_{M,d} : S_2(M) \to S_2(\Gamma_0(N))$ given by $\beta_{M,d}(f(q)) = f(q^d)$. The *new subspace* of $S_2(\Gamma_0(N))$, denoted $S_2(\Gamma_0(N))_{\mathrm{new}}$, is the complementary $\mathbb{T}$-submodule of the $\mathbb{T}$-module generated by the images of all maps $\beta_{M,d}$, with M and d as above. (It is a nontrivial fact that this complement is well defined; one possible proof uses the Petersson inner product.)

The theory of Atkin and Lehner [AL70] (see Section 6.1.1) asserts that, as a $\mathbb{T}'$-module, $S_2(\Gamma_0(N))$ decomposes as follows:

$$S_2(\Gamma_0(N)) \quad = \bigoplus_{M \mid N,\ d \mid N/M} \beta_{M,d}(S_2(M)_{\mathrm{new}}).$$

To compute $S_2(\Gamma_0(N))$ it thus suffices to compute $S_2(M)_{\mathrm{new}}$ for each positive divisor M of N.

We now turn to the problem of computing $S_2(\Gamma_0(N))_{\mathrm{new}}$. Atkin and Lehner [AL70] also proved that $S_2(\Gamma_0(N))_{\mathrm{new}}$ is spanned by eigenforms, each of which occurs with multiplicity one in $S_2(\Gamma_0(N))_{\mathrm{new}}$. Moreover, if $f \in S_2(\Gamma_0(N))_{\mathrm{new}}$ is an eigenform then the coefficient of q in the q-expansion of f is nonzero, so it is possible to normalize f so that coefficient of q is 1. With f so normalized, if $T_p(f) = a_p f$, then the pth Fourier coefficient of f is a_p. If $f = \sum_{n=1}^{\infty} a_n q^n$ is a normalized eigenvector for all T_p, then the a_n, with n composite, are determined by the a_p, with p prime, by the following formulas: $a_{nm} = a_n a_m$ when n and m are relatively prime, and $a_{p^r} = a_{p^{r-1}} a_p - p a_{p^{r-2}}$ for $p \nmid N$ prime. When $p \mid N$, $a_{p^r} = a_p^r$. We conclude that in order to compute $S_2(\Gamma_0(N))_{\mathrm{new}}$, it suffices to compute all systems of eigenvalues $\{a_2, a_3, a_5, \ldots\}$ of the Hecke operators $T_2, T_3, T_5, \ldots$ acting on $S_2(\Gamma_0(N))_{\mathrm{new}}$. Given a system of eigenvalues, the corresponding eigenform is $f = \sum_{n=1}^{\infty} a_n q^n$, where the a_n, for n composite, are determined by the recurrence given above.

In light of the pairing $\langle\, , \rangle$ introduced in Section 3.1, computing the above systems of eigenvalues $\{a_2, a_3, a_5, \ldots\}$ amounts to computing the systems of eigenvalues of the Hecke operators T_p on the subspace V of $\mathbb{S}_2(\Gamma_0(N))$ that corresponds to the new subspace of $S_2(\Gamma_0(N))$. For each proper divisor M of N

and each divisor d of N/M, let $\phi_{M,d} : \mathbb{S}_2(\Gamma_0(N)) \to \mathbb{S}_2(\Gamma_0(M))$ be the map sending x to $\left(\begin{smallmatrix} d & 0 \\ 0 & 1 \end{smallmatrix}\right) x$. Then V is the intersection of the kernels of all maps $\phi_{M,d}$.

The computation of the systems of eigenvalues of a collection of commuting diagonalizable endomorphisms involves standard linear algebra techniques, such as computation of characteristic polynomials and kernels of matrices. There are, however, several tricks that greatly speed up this process, some of which are described in Chapter 7.

Example 3.7.1. All forms in $S_2(\Gamma_0(39))$ are new. Up to Galois conjugacy, the eigenvalues of the Hecke operators T_2, T_3, T_5, and T_7 on $\mathbb{S}_2(\Gamma_0(39))$ are $\{1, -1, 2, -4\}$ and $\{a, 1, -2a-2, 2a+2\}$, where $a^2 + 2a - 1 = 0$. Each of these eigenvalues occur in $\mathbb{S}_2(\Gamma_0(39))$ with multiplicity two; for example, the characteristic polynomial of T_2 on $\mathbb{S}_2(\Gamma_0(39))$ is $(x-1)^2 \cdot (x^2+2x-1)^2$. Thus $S_2(\Gamma_0(39))$ is spanned by

$$\begin{aligned}
f_1 &= q + q^2 - q^3 - q^4 + 2q^5 - q^6 - 4q^7 + \cdots, \\
f_2 &= q + aq^2 + q^3 + (-2a-1)q^4 + (-2a-2)q^5 + aq^6 + (2a+2)q^7 + \cdots, \\
f_3 &= q + \sigma(a)q^2 + q^3 + (-2\sigma(a)-1)q^4 + (-2\sigma(a)-2)q^5 + \sigma(a)q^6 + (2\sigma(a)+2)q^7 + \cdots,
\end{aligned}$$

where $\sigma(a)$ is the $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-conjugate of a.

3.7.1 Summary

To compute the q-expansion of each eigenforms in $S_2(\Gamma_0(N))$, we use the degeneracy maps so that we only have to solve the problem for $S_2(\Gamma_0(N))_{\mathrm{new}}$. Using modular symbols, we compute all systems of eigenvalues $\{a_2, a_3, a_5, \ldots\}$, then write down the corresponding eigenforms $f = q + a_2q^2 + a_3q^3 + \cdots$.