3.8 Exercises

3.1 Suppose that $\lambda, \lambda' \in \mathfrak{h}$ are in the same orbit for the action of $\Gamma_0(N)$, i.e., that there exists $g \in \Gamma_0(N)$ such that $g(\lambda) = \lambda'$. Let $\Lambda = \mathbb{Z}+\mathbb{Z}\lambda$ and $\Lambda' = \mathbb{Z} + \mathbb{Z}\lambda'$. Prove that the pairs $(\mathbb{C}/\Lambda, (1/N)\mathbb{Z}/\Lambda)$ and $(\mathbb{C}/\Lambda', (1/N)\mathbb{Z}/\Lambda')$ are isomorphic. (By an isomorphism $(E, C) \to (F, D)$ of pairs, we mean an isomorphism $\phi : E \to F$ of elliptic curves that sends C to D. You may use the fact that an isomorphic of elliptic curves over $\mathbb{C}$ is a $\mathbb{C}$-linear $\mathbb{C} \to \mathbb{C}$ that sends the lattice corresponding to one curve onto the lattice corresponding to the other.)

3.2 Prove that for any integers n, m and any level N, the modular symbol $\{n, m\}$ is 0 as an element of $\mathbb{M}_2(\Gamma_0(N))$. [Hint: See Example 3.2.1.]

3.3 Let p be a prime.

(a) List representative elements of $\mathbb{P}^1(\mathbb{Z}/3\mathbb{Z})$.

(b) What is the cardinality of $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$ as a function of p?

(c) Prove that there is a bijection between the right cosets of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{Z})$ and the elements of $\mathbb{P}^1(\mathbb{Z}/p\mathbb{Z})$. (As mentioned in this chapter this is also true for composite level; see [Cre97a, §2.2] for complete details.)

3.4 Use the inductive proof of Proposition 3.3.2 to write $\{0, 4/7\}$ in terms of Manin symbols for $\Gamma_0(7)$.

3.5 Show that the Hecke operator T_2 acts as multiplication by 3 on the space $\mathbb{M}_2(\Gamma_0(3))$ as follows:

(a) Write down right coset representatives for $\Gamma_0(3)$ in $\mathrm{SL}_2(\mathbb{Z})$.

(b) List all 8 relations coming from 3.3.4.

(c) Find a single Manin symbols $[r_i]$ so that the three other Manin symbols are a nonzero multiple of $[r_i]$ modulo the relations found in the previous step.

(d) Use formula (3.4.1) to compute the image of your symbol $[r_i]$ under T_2. You will obtain a sum of four symbols. Using the relations above, write this sum as a multiple of $[r_i]$. (The multiple must be 3 or you made a mistake.)