Lecture 34: The Birch and Swinnerton-Dyer Conjecture, Part 1

William Stein

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The next three lectures will be about the Birch and Swinnerton-Dyer conjecture, which is considered by many people to be the most important accessible open problem in number theory. Today I will guide you through Wiles's Clay Math Institute paper on the Birch and Swinnerton-Dyer conjecture.

On Friday, I will talk about the following open problem, which is a frustrating specific case of the Birch and Swinnerton-Dyer conjecture. Let E be the elliptic curve defined by

$$y^2 + xy = x^3 - x^2 - 79x + 289.$$

Denote by $L(E,s) = \sum_{n=1}^{\infty} a_n n^{-s}$ the corresponding L-series, which extends to a function everywhere. The graph of $L(E,s)$ for $s \in (0,5)$ is given on the next page. It can be proved that $E(\mathbb{Q}) \approx \mathbb{Z}^4$ by showing that

$$(8,7),\ \left(\frac{120}{27}, \frac{29}{27}\right),\ \left(\frac{70}{8}, \frac{81}{8}\right),\ \text{and}\ \left(\frac{564}{8}, \frac{665}{64}\right)$$

generate a "subgroup of finite index" in $E(\mathbb{Q})$. The Birch and Swinnerton-Dyer Conjecture then predicts that

$$\operatorname{ord}_{s=1} L(E,s) = 4,$$

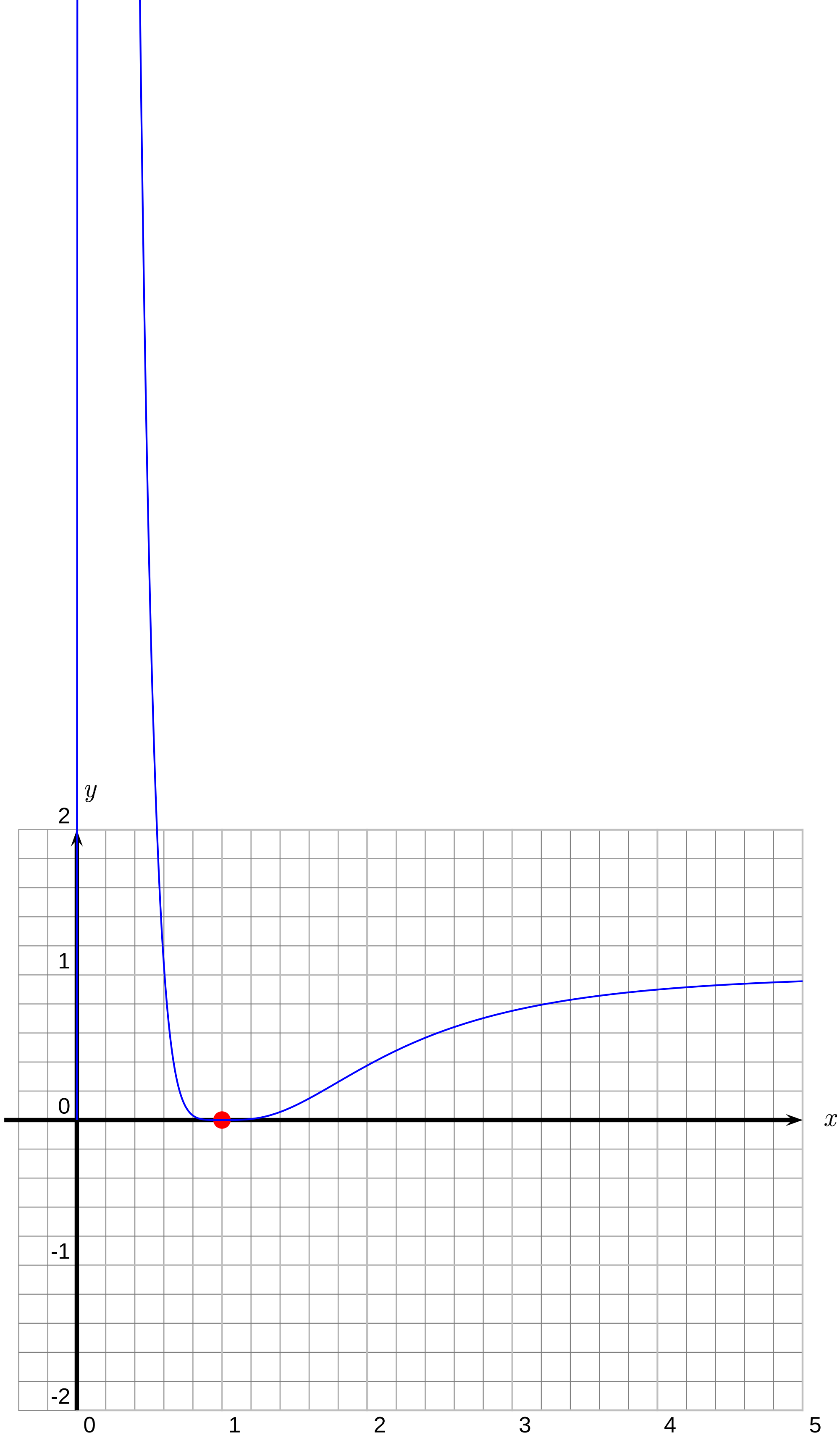
which looks plausible from the shape of the graph on the next page. It is relatively easy to prove that the following is equivalent to showing that $\operatorname{ord}_{s=1} L(E,s) = 4$:

Open Problem: *Prove that* $L''(E,1) = 0$.

If you could solve this open problem, people like Gross, Tate, Mazur, Zagier, Wiles, me, etc., would be **very** excited. The related problem of giving an example of an L-series with $\operatorname{ord}_{s=1} L(E,s) = 3$, was solved as a consequence of a very deep theorem of Gross and Zagier, and resulting in an effective solution to Gauss's class number problem.

John Tate gave a talk about the BSD conjecture for the Clay Math Institute. I strongly encourage you to watch it online at

`http://www.msri.org/publications/ln/hosted/cmi/2000/cmiparis/index-tate.html`



The L-series of the "simplest" known elliptic curve of rank 4.