Lecture 17: Continued Fractions, I

William Stein

Math 124 HARVARD UNIVERSITY **Fall 2001**

1 Introduction

A *continued fraction* is an expression of the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}$$

which may or may not go on indefinitely. We denote[1] the value of this continued fraction by

$$[a_0, a_1, a_2, \ldots].$$

The a_n are called the *partial quotients* of the continued fraction (we will see why at the end of this lecture). Thus, e.g.,

$$[1, 2] = 1 + \frac{1}{2} = \frac{3}{2},$$

and

$$\frac{172}{51} = [3, 2, 1, 2, 6] = 3 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{2 + \frac{1}{6}}}}.$$

Continued fractions have many applications, from the abstract to the concrete. They give good rational approximations to irrational numbers, and the have been used to understand why you can't tune a piano perfectly.[2] Continued fractions also suggest a sense in which e appears to be "less transcendental" than π.

There are many places to read about continued fractions, including Chapter X of Hardy and Wright's *Intro. to the Theory of Numbers*, §13.3 of Burton's *Elementary Number Theory*, Chapter IV of Davenport, and Khintchine's *Continued Fractions*. The notes you're reading right now draw primarily on Hardy and Wright, since their exposition is very clear and to the point. I found Davenport's chapter IV uneccessarily tedious; I felt marched through a thick jungle to see a beautiful river.

[1] *Warning:* This notation clashes with the notation used in Davenport. Our notation is standard.
[2] See `http://www.research.att.com/~njas/sequences/DUNNE/TEMPERAMENT.HTML`

2 Finite Continued Fractions

Definition 2.1. A *finite continued fraction* is an expression

$$a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \cdots + \frac{1}{a_m}}},$$

where each a_n is a rational number and $a_n > 0$ for all $n \geq 1$. If the a_n are integers, we say that the continued fraction is *integral.*

To get a feeling for continued fractions, observe that

$$\begin{aligned}
[a_0] &= a_0, \\
[a_0, a_1] &= a_0 + \frac{1}{a_1} = \frac{a_0 a_1 + 1}{a_1}, \\
[a_0, a_1, a_2] &= a_0 + \frac{1}{a_1 + \frac{1}{a_2}} = \frac{a_0 a_1 a_2 + a_0 + a_2}{a_1 a_2 + 1}.
\end{aligned}$$

Also,

$$\begin{aligned}
[a_0, a_1, \ldots, a_{m-1}, a_m] &= [a_0, a_1, \ldots, a_{m-2}, a_{m-1} + \frac{1}{a_m}] \\
&= a_0 + \frac{1}{[a_1, \ldots, a_m]} \\
&= [a_0, [a_1, \ldots, a_m]].
\end{aligned}$$

2.1 Partial Convergents

Fix a continued fraction $[a_0, \ldots, a_m]$.

Definition 2.2. For $0 \leq n \leq m$, the nth *convergent* of the continued fraction $[a_0, \ldots, a_m]$ is $[a_0, \ldots, a_n]$.

For each $n \geq -1$, define real numbers p_n and q_n as follows:

$$\begin{aligned}
p_{-1} &= 1, & p_0 &= a_0, & p_1 &= a_1 p_0 + p_{-1} = a_1 a_0 + 1, & p_n &= a_n p_{n-1} + p_{n-2}, \\
q_{-1} &= 0, & q_0 &= 1, & q_1 &= a_1 q_0 + q_{-1} = a_1, & q_n &= a_n q_{n-1} + q_{n-2}.
\end{aligned}$$

Exercise 2.3. [3] Compute p_n and q_n for the continued fractions $[-3, 1, 1, 1, 1, 3]$ and $[0, 2, 4, 1, 8, 2]$. Observe that the propositions below hold.

Proposition 2.4. $[a_0, \ldots, a_n] = \dfrac{p_n}{q_n}$

[3] Try to do this exercise, which is not part of the regular homework, before the next lecture.

Proof. We use induction. We already verified the assertion when $n = 0, 1$. Suppose the proposition is true for all continued fractions of length $n-1$. Then

$$\begin{aligned}
[a_0, \ldots, a_n] &= [a_0, \ldots, a_{n-2}, a_{n-1} + \frac{1}{a_n}] \\
&= \frac{\left(a_{n-1} + \frac{1}{a_n}\right) p_{n-2} + p_{n-3}}{\left(a_{n-1} + \frac{1}{a_n}\right) q_{n-2} + q_{n-3}} \\
&= \frac{(a_{n-1}a_n + 1)p_{n-2} + a_n p_{n-3}}{(a_{n-1}a_n + 1)q_{n-2} + a_n q_{n-3}} \\
&= \frac{a_n(a_{n-1}p_{n-2} + p_{n-3}) + p_{n-2}}{a_n(a_{n-1}q_{n-2} + q_{n-3}) + q_{n-2}} \\
&= \frac{a_n p_{n-1} + p_{n-2}}{a_n q_{n-1} + q_{n-2}} = \frac{p_n}{q_n}.
\end{aligned}$$

□

Proposition 2.5. *For $n \leq m$,*

1. *the determinant of* $\begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}$ *is* $(-1)^{n-1}$*; equivalently,*
$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}};$$
2. *the determinant of* $\begin{pmatrix} p_n & p_{n-2} \\ q_n & q_{n-2} \end{pmatrix}$ *is* $(-1)^n a_n$*; equivalently,*
$$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}}.$$

Proof. For the first statement, we proceed by induction. The case $n = 0$ holds because the determinant of $\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix}$ is $-1 = (-1)^{-1}$. Suppose the statement is true for $n-1$. Then

$$\begin{aligned}
p_n q_{n-1} - q_n p_{n-1} &= (a_n p_{n-1} + p_{n-2})q_{n-1} - (a_n q_{n-1} + q_{n-2})p_{n-1} \\
&= p_{n-2}q_{n-1} - q_{n-2}p_{m-1} \\
&= -(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}) \\
&= -(-1)^{n-2} = (-1)^{n-1}.
\end{aligned}$$

This completes the proof of the first statement. For the second statement,

$$\begin{aligned}
p_n q_{n-2} - p_{n-2} q_n &= (a_n p_{n-1} + p_{n-2})q_{n-2} - p_{n-2}(a_n q_{n-1} + q_{n-2}) \\
&= a_n(p_{n-1}q_{n-2} - p_{n-2}q_{n-1}) \\
&= (-1)^n a_n.
\end{aligned}$$

□

Corollary 2.6. *The fraction* $\frac{p_n}{q_n}$ *is in lowest terms.*

Proof. If $p \mid p_n$ and $p \mid q_n$ then $p \mid (-1)^{n-1}$. □

2.2 How the Convergents Converge

Let $[a_0, \ldots, a_m]$ be a continued fraction and for $n \leq m$ let

$$c_n = [a_0, \ldots, a_n] = \frac{p_n}{q_n}$$

denote the nth convergent.

Proposition 2.7. *The even convergents c_{2n} increase strictly with n, and the odd convergents c_{2n+1} decrease strictly with n. Moreover, the odd convergents c_{2n+1} are greater than all of the even convergents.*

Proof. For $n \geq 1$ the a_n are positive, so the q_n are all positive. By Proposition 2.5, for $n \geq 2$,

$$c_n - c_{n-2} = (-1)^n \cdot \frac{a_n}{q_n q_{n-2}},$$

which proves the first claim.

Next, Proposition 2.5 implies that for $n \geq 1$,

$$c_n - c_{n-1} = (-1)^{n-1} \cdot \frac{1}{q_n q_{n-1}}$$

has the sign of $(-1)^{n-1}$, so that $c_{2n+1} > c_{2n}$. Thus if there exists r, n such that $c_{2n+1} < c_{2r}$, then $r \neq n$. If $r < n$, then $c_{2n+1} < c_{2r} < c_{2n}$, a contradiction. If $r > n$, then $c_{2r+1} < c_{2n+1} < c_{2r}$, also a contradiction. ☐

3 Every Rational Number is Represented

Proposition 3.1. *Every rational number is represented by a continued fraction.*

Proof. Let a/b, where $b > 0$, be any rational number. Euclid's algorithm gives:

$$\begin{aligned} a &= b \cdot a_0 + r_1, & 0 < r_1 < b \\ b &= r_1 \cdot a_1 + r_2, & 0 < r_2 < r_1 \\ &\cdots \\ r_{n-2} &= r_{n-1} \cdot a_{n-1} + r_n, & 0 < r_n < r_{n-1} \\ r_{n-1} &= r_n \cdot a_n + 0. \end{aligned}$$

Note that $a_i > 0$ for $i > 0$. Rewrite the equations as follows:

$$\begin{aligned} a/b &= a_0 + r_1/b = a_0 + 1/(b/r_1), \\ b/r_1 &= a_1 + r_2/r_1 = a_1 + 1/(r_1/r_2), \\ r_1/r_2 &= a_2 + r_3/r_2 = a_2 + 1/(r_2/r_3), \\ &\cdots \\ r_{n-1}/r_n &= a_n. \end{aligned}$$

It follows that

$$\frac{a}{b} = [a_0, a_1, \ldots a_n].$$

☐