

is the space of holomorphic functions $f : \mathfrak{h} \rightarrow \mathbf{C}$ that satisfy the usual vanishing conditions at the cusps and such that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(M, N)$,

$$f \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. = \varepsilon(d)f.$$

We have

$$S_k(M, N) = \bigoplus_{\varepsilon} S_k(M, N, \varepsilon).$$

We now introduce operators between various $S_k(M, N)$. Note that, except when otherwise noted, the notation we use for these operators below is as in [Li75], which conflicts with notation in various other books. When in doubt, check the definitions.

Let

$$f \left| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right. (\tau) = (ad - bc)^{k/2} (c\tau + d)^{-k} f \left(\frac{a\tau + b}{c\tau + d} \right).$$

This is like before, but we omit the weight k from the bar notation, since k will be fixed for the whole discussion.

For any d and $f \in S_k(M, N, \varepsilon)$, define

$$f|U_d^N = d^{k/2-1} f \left| \left(\sum_{u \bmod d} \begin{pmatrix} 1 & uN \\ 0 & d \end{pmatrix} \right) \right.,$$

where the sum is over *any set* u of representatives for the integers modulo d . Note that the N in the notation is a superscript, not a power of N . Also, let

$$f|B_d = d^{-k/2} f \left| \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \right.,$$

and

$$f|C_d = d^{k/2} f \left| \begin{pmatrix} 1 & 0 \\ 0 & d \end{pmatrix} \right..$$

In [Li75], C_d is denoted W_d , which would be confusing, since in the literature W_d is usually used to denote a completely different operator (the Atkin-Lehner operator, which is denoted V_d^M in [Li75]).

Since $\begin{pmatrix} 1 & N \\ 0 & 1 \end{pmatrix} \in \Gamma(M, N)$, any $f \in S_k(M, N, \varepsilon)$ has a Fourier expansion in terms of powers of $q_N = q^{1/N}$. We have

$$\left(\sum a_n q_N^n \right) |U_d^N = \sum_{n \geq 1} a_{nd} q_N^n,$$

$$\left(\sum a_n q_N^n \right) |B_d = \sum_{n \geq 1} a_n q_N^{nd},$$

and

$$\left(\sum a_n q_N^n \right) |C_d = \sum_{n \geq 1} a_n q_N^n.$$

The second two equalities are easy to see; for the first, write everything out and use that for $n \geq 1$, the sum $\sum_u e^{2\pi i u n/d}$ is 0 or d if $d \nmid n$, $d \mid n$, respectively.

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The maps B_d and C_d define injective maps between various spaces $S_k(M, N, \varepsilon)$. To understand B_d , use the matrix relation

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$$\begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} x & dy \\ z/d & w \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix},$$

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and a similar one for C_d . If $d \mid N$ then $B_d : S_k(M, N, \varepsilon) \rightarrow S_k(dM, N/d, \varepsilon)$ is an isomorphism, and if $d \mid M$, then $C_d : S_k(M, N) \rightarrow S_k(M/d, Nd, \varepsilon)$ is also an isomorphism. In particular, taking $d = N$, we obtain an isomorphism

$$B_N : S_k(M, N, \varepsilon) \rightarrow S_k(MN, 1, \varepsilon) = S_k(\Gamma_1(MN), \varepsilon). \quad (9.1.1)$$

Putting these maps together allows us to completely understand the cusp forms $S_k(\Gamma(N))$ in terms of spaces $S_k(\Gamma_1(N^2), \varepsilon)$, for all Dirichlet characters ε that arise from characters modulo N . (Recall that $\Gamma(N)$ is the principal congruence subgroup $\Gamma(N) = \ker(\mathrm{SL}_2(\mathbf{Z}) \rightarrow \mathrm{SL}_2(\mathbf{Z}/N\mathbf{Z}))$. This is because $S_k(\Gamma(N))$ is isomorphic to the direct sum of $S_k(N, N, \varepsilon)$, as ε varies over all Dirichlet characters modulo N .

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For any prime p , the p th Hecke operator on $S_k(M, N, \varepsilon)$ is defined by

$$T_p = U_p^N + \varepsilon(p)p^{k-1}B_p.$$

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Note that $T_p = U_p^N$ when $p \mid N$, since then $\varepsilon(p) = 0$. In terms of Fourier expansions, we have

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$$\left(\sum a_n q_N^n \right) | T_p = \sum_{n \geq 1} (a_{np} + \varepsilon(p)p^{k-1}a_{n/p}) q_N^n,$$

where $a_{n/p} = 0$ if $p \nmid n$.

The operators we have just defined satisfy several commutativity relations. Suppose p and q are prime. Then $T_p B_q = B_q T_p$, $T_p C_q = C_q T_p$, and $T_p U_q^N = U_q^N T_p$ if $(p, qMN) = 1$. Moreover $U_d^N B_{d'} = B_{d'} U_d^N$ if $(d, d') = 1$.

Remark 9.1.1. Because of these relations, (9.1.1) describe $S_k(\Gamma(N))$ as a module over the ring generated by T_p for $p \nmid N$.

Definition 9.1.2 (Old Subspace). The *old subspace* $S_k(M, N, \varepsilon)_{\mathrm{old}}$ is the subspace of $S_k(M, N, \varepsilon)$ generated by all $f|B_d$ and $g|C_e$ where $f \in S_k(M', N)$, $g \in S_k(M, N')$, and M', N' are proper factors of M, N , respectively, and $d \mid M/M'$, $e \mid N/N'$.

Since T_p commutes with B_d and C_e , the Hecke operators T_p all preserve $S_k(M, N, \varepsilon)_{\mathrm{old}}$, for $p \nmid MN$. Also, B_N defines an isomorphism

$$S_k(M, N, \varepsilon)_{\mathrm{old}} \cong S_k(MN, 1, \varepsilon)_{\mathrm{old}}.$$

Definition 9.1.3 (Petersson Inner Product). If $f, g \in S_k(\Gamma(N))$, the *Petersson inner product* of f and g is

$$\langle f, g \rangle = \frac{1}{[\mathrm{SL}_2(\mathbf{Z}) : \Gamma(N)]} \int_D f(z) \overline{g(z)} y^{k-2} dx dy,$$

where D is a fundamental domain for $\Gamma(N)$ and $z = x + iy$.

This Petersson pairing is normalized so that if we consider f and g as elements of $\Gamma(N')$ for some multiple N' of N , then the resulting pairing is the same (since the volume of the fundamental domain shrinks by the index).

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Proposition 9.1.4 (Petersson). *If $p \nmid N$ and $f \in S_k(\Gamma_1(N), \varepsilon)$, then $\langle f|T_p, g \rangle = \varepsilon(p)\langle f, g|T_p \rangle$.*

Remark 9.1.5. The proposition implies that the T_p , for $p \nmid N$, are diagonalizable. Be careful, because the T_p , with $p \mid N$, need not be diagonalizable.

Definition 9.1.6 (New Subspace). The *new subspace* $S_k(M, N, \varepsilon)_{\text{new}}$ is the orthogonal complement of $S_k(M, N, \varepsilon)_{\text{old}}$ in $S_k(M, N, \varepsilon)$ with respect to the Petersson inner product.

Both the old and new subspaces of $S_k(M, N, \varepsilon)$ are preserved by the Hecke operators T_p with $(p, NM) = 1$.

Remark 9.1.7. Li [Li75] also gives a purely algebraic definition of the new subspace as the intersection of the kernels of various trace maps from $S_k(M, N, \varepsilon)$, which are obtained by averaging over coset representatives.

Definition 9.1.8 (Newform). A *newform* $f = \sum a_n q_N^n \in S_k(M, N, \varepsilon)$ is an element of $S_k(M, N, \varepsilon)_{\text{new}}$ that is an eigenform for all T_p , for $p \nmid NM$, and is normalized so that $a_1 = 1$.

Li introduces the crucial “Atkin-Lehner operator” W_q^M (denoted V_q^M in [Li75]), which plays a key roll in all the proofs, and is defined as follows. For a positive integer M and prime q , let $\alpha = \text{ord}_q(M)$ and find integers x, y, z such that $q^{2\alpha}x - yMz = q^\alpha$. Then W_q^M is the operator defined by slashing with the matrix $\begin{pmatrix} q^\alpha x & y \\ Mz & q^\alpha \end{pmatrix}$. Li shows that if $f \in S_k(M, 1, \varepsilon)$, then $f|W_q^M|W_q^M = \varepsilon(q^\alpha)f$, so W_q^M is an automorphism. Care must be taken, because the operator W_q^M need not commute with $T_p = U_p^N$, when $p \mid M$.

After proving many technical but elementary lemmas about the operators B_d , C_d , U_p^N , T_p , and W_q^M , Li uses the lemmas to deduce the following theorems. The proofs are all elementary, but there is **little I can say about them**, except that you just have to read them.¹

Theorem 9.1.9. *Suppose $f = \sum a_n q_N^n \in S_k(M, N, \varepsilon)$ and $a_n = 0$ for all n with $(n, K) \neq 1$, where K is a fixed positive integer. Then $f \in S_k(M, N, \varepsilon)_{\text{old}}$.*

From the theorem we see that if f and g are newforms in $S_k(M, N, \varepsilon)$, and if for all but finitely many primes p , the T_p eigenvalues of f and g are the same, then $f - g$ is an old form, so $f - g = 0$, hence $f = g$. Thus the eigenspaces corresponding to the systems of Hecke eigenvalues associated to the T_p , with $p \nmid MN$, each have dimension 1. This is known as “multiplicity one”.

Theorem 9.1.10. Let $f = \sum a_n q_N^n$ be a newform in $S_k(M, N, \varepsilon)$, p a prime with $(p, MN) = 1$, and $q \mid MN$ a prime. Then

$$1. f|T_p = a_p f, f|U_q^N = a_q f, \text{ and for all } n \geq 1,$$

$$a_p a_n = a_{np} + \varepsilon(p)p^{k-1} a_{n/p},$$

$$a_q a_n = a_{nq}.$$

¹Remove from book.

If $L(f, s) = \sum_{n \geq 1} a_n n^{-s}$ is the Dirichlet series associated to f , then $L(f, s)$ has an Euler product

$$L(f, s) = \prod_{q|MN} (1 - a_q q^{-s})^{-1} \prod_{p \nmid MN} (1 - a_p p^{-s} + \varepsilon(p) p^{k-1} p^{-2s})^{-1}.$$

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2. (a) If ε is not a character mod MN/q , then $|a_q| = q^{(k-1)/2}$.
- (b) If ε is a character mod MN/q , then $a_q = 0$ if $q^2 \nmid MN$, and $a_q^2 = \varepsilon(q)q^{k-2}$ if $q^2 \mid MN$.

9.2 The U_p operator

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Let N be a positive integer and M a divisor of N . For each divisor d of N/M we define a map

$$\alpha_d : S_k(\Gamma_1(M)) \rightarrow S_k(\Gamma_1(N)) : f(\tau) \mapsto f(d\tau).$$

We verify that $f(d\tau) \in S_k(\Gamma_1(N))$ as follows. Recall that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we write

$$(f|[\gamma]_k)(\tau) = \det(\gamma)^{k-1} (cz + d)^{-k} f(\gamma(\tau)).$$

The transformation condition for f to be in $S_k(\Gamma_1(N))$ is that $f|[\gamma]_k(\tau) = f(\tau)$. Let $f(\tau) \in S_k(\Gamma_1(M))$ and let $\iota_d = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix}$. Then $f|[\iota_d]_k(\tau) = d^{k-1} f(d\tau)$ is a modular form on $\Gamma_1(N)$ since $\iota_d^{-1} \Gamma_1(M) \iota_d$ contains $\Gamma_1(N)$. Moreover, if f is a cusp form then so is $f|[\iota_d]_k$.

Proposition 9.2.1. *If $f \in S_k(\Gamma_1(M))$ is nonzero, then*

$$\left\{ \alpha_d(f) : d \mid \frac{N}{M} \right\}$$

is linearly independent.

Proof. If the q -expansion of f is $\sum a_n q^n$, then the q -expansion of $\alpha_d(f)$ is $\sum a_n q^{dn}$. The matrix of coefficients of the q -expansions of $\alpha_d(f)$, for $d \mid (N/M)$, is upper triangular. Thus the q -expansions of the $\alpha_d(f)$ are linearly independent, hence the $\alpha_d(f)$ are linearly independent, since the map that sends a cusp form to its q -expansion is linear. \square

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When $p \mid N$, we denote by U_p the Hecke operator T_p acting on the image space $S_k(\Gamma_1(N))$. For clarity, in this section we will denote by $T_{p,M}$, the Hecke operator $T_p \in \text{End}(S_k(\Gamma_1(M)))$. For $f = \sum a_n q^n \in S_k(\Gamma_1(N))$, we have

$$f|U_p = \sum a_{np} q^n.$$

Suppose $f = \sum a_n q^n \in S_k(\Gamma_1(M))$ is a normalized eigenform for all of the Hecke operators T_n and $\langle n \rangle$, and p is a prime that does not divide M . Then

$$f|T_{p,M} = a_p f \quad \text{and} \quad f| \langle p \rangle = \varepsilon(p) f.$$

Assume $N = p^r M$, where $r \geq 1$ is an integer. Let

$$f_i(\tau) = f(p^i \tau),$$

so f_0, \dots, f_r are the images of f under the maps $\alpha_{p^0}, \dots, \alpha_{p^r}$, respectively, and $f = f_0$. We have

$$\begin{aligned} f|T_{p,M} &= \sum_{n \geq 1} a_n q^n + \varepsilon(p) p^{k-1} \sum a_n q^{pn} \\ &= f_0|U_p + \varepsilon(p) p^{k-1} f_1, \end{aligned}$$

so

$$f_0|U_p = f|T_{p,M} - \varepsilon(p) p^{k-1} f_1 = a_p f_0 - \varepsilon(p) p^{k-1} f_1.$$

Also

$$f_1|U_p = \left(\sum a_n q^{pn} \right) |U_p = \sum a_n q^n = f_0.$$

More generally, for any $i \geq 1$, we have $f_i|U_p = f_{i-1}$.

The operator U_p preserves the two dimensional vector space spanned by f_0 and f_1 , and the matrix of U_p with respect to the basis f_0, f_1 is

$$A = \begin{pmatrix} a_p & 1 \\ -\varepsilon(p) p^{k-1} & 0 \end{pmatrix},$$

which has characteristic polynomial

$$X^2 - a_p X + p^{k-1} \varepsilon(p). \quad (9.2.1)$$

9.2.1 A Connection with Galois representations

This leads to a striking connection with Galois representations. Let f be a newform and let $K = K_f$ be the field generated over \mathbf{Q} by the Fourier coefficients of f . Let ℓ be a prime and λ a prime lying over ℓ . Then Deligne (and Serre, when $k = 1$) constructed a representation

$$\rho_\lambda : \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \rightarrow \text{GL}(2, K_\lambda).$$

If $p \nmid N\ell$, then ρ_λ is unramified at p , so if $\text{Frob}_p \in \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ is a Frobenius element, then $\rho_\lambda(\text{Frob}_p)$ is well defined, up to conjugation. Moreover, ~~one can show that~~

$$\begin{aligned} \det(\rho_\lambda(\text{Frob}_p)) &= p^{k-1} \varepsilon(p), \quad \text{and} \\ \text{tr}(\rho_\lambda(\text{Frob}_p)) &= a_p. \end{aligned}$$

(We will discuss the proof of these relations further in the case $k = 2$.) Thus the characteristic polynomial of $\rho_\lambda(\text{Frob}_p) \in \text{GL}_2(E_\lambda)$ is

$$X^2 - a_p X + p^{k-1} \varepsilon(p),$$

which is the same as (9.2.1).

9.2.2 When is U_p semisimple?

Question 9.2.2. Is U_p semisimple on the span of f_0 and f_1 ?

If the eigenvalues of U_p are distinct, then the answer is yes. If the eigenvalues are the same, then $X^2 - a_p X + p^{k-1}\varepsilon(p)$ has discriminant 0, so $a_p^2 = 4p^{k-1}\varepsilon(p)$, hence

$$a_p = 2p^{\frac{k-1}{2}} \sqrt{\varepsilon(p)}.$$

Open Problem 9.2.3. Does there exist an eigenform $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ such that $a_p = 2p^{\frac{k-1}{2}} \sqrt{\varepsilon(p)}$?

It is a curious fact that the Ramanujan conjectures, which were proved by Deligne in 1973, imply that $|a_p| \leq 2p^{(k-1)/2}$, so the above equality remains taunting. When $k = 2$, Coleman and Edixhoven proved that $|a_p| < 2p^{(k-1)/2}$.

9.2.3 An Example of non-semisimple U_p

Suppose $f = f_0$ is a normalized eigenform. Let W be the space spanned by f_0, f_1 and let V be the space spanned by f_0, f_1, f_2, f_3 . Then U_p acts on V/W by $\bar{f}_2 \mapsto 0$ and $\bar{f}_3 \mapsto \bar{f}_2$. Thus the matrix of the action of U_p on V/W is $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, which is nonzero and nilpotent, hence not semisimple. Since W is invariant under U_p this shows that U_p is not semisimple on V , i.e., U_p is not diagonalizable.

9.3 The Cusp forms are free of rank one over $\mathbf{T}_{\mathbf{C}}$

9.3.1 Level 1

Suppose $N = 1$, so $\Gamma_1(N) = \text{SL}_2(\mathbf{Z})$. Using the Petersson inner product, we see that all the T_n are diagonalizable, so $S_k = S_k(\Gamma_1(1))$ has a basis

$$f_1, \dots, f_d$$

of normalized eigenforms where $d = \dim S_k$. This basis is canonical up to ordering. Let $\mathbf{T}_{\mathbf{C}} = \mathbf{T} \otimes \mathbf{C}$ be the ring generated over \mathbf{C} by the Hecke operator T_p . Then, having fixed the basis above, there is a canonical map

$$\mathbf{T}_{\mathbf{C}} \hookrightarrow \mathbf{C}^d : T \mapsto (\lambda_1, \dots, \lambda_d),$$

where $f_i|T = \lambda_i f_i$. This map is injective and $\dim \mathbf{T}_{\mathbf{C}} = d$, so the map is an isomorphism of \mathbf{C} -vector spaces.

The form

$$v = f_1 + \dots + f_n$$

generates S_k as a \mathbf{T} -module. Note that v is canonical since it does not depend on the ordering of the f_i . Since v corresponds to the vector $(1, \dots, 1)$ and $\mathbf{T} \cong \mathbf{C}^d$

²Look in Coleman-edixhoven and say more about this. Plus find the Weil reference. When $k = 2$, Weil [?] showed that $\rho_\lambda(\text{Frob}_p)$ is semisimple, so if the eigenvalues of U_p are equal then $\rho_\lambda(\text{Frob}_p)$ is a scalar. But Edixhoven and Coleman [CE98] show that it is not a scalar by looking at the abelian variety attached to f .

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acts on $S_k \cong \mathbf{C}^d$ componentwise, this is just the statement that \mathbf{C}^d is generated by $(1, \dots, 1)$ as a \mathbf{C}^d -module.

There is a perfect pairing $S_k \times \mathbf{T}_{\mathbf{C}} \rightarrow \mathbf{C}$ given by

$$\left\langle \sum f, T_n \right\rangle = a_1(f|T_n) = a_n(f),$$

where $a_n(f)$ denotes the n th Fourier coefficient of f . Thus we have simultaneously:

1. S_k is free of rank 1 over $\mathbf{T}_{\mathbf{C}}$, and
2. $S_k \cong \text{Hom}_{\mathbf{C}}(\mathbf{T}_{\mathbf{C}}, \mathbf{C})$ as \mathbf{T} -modules.

Combining these two facts yields an isomorphism

$$\mathbf{T}_{\mathbf{C}} \cong \text{Hom}_{\mathbf{C}}(\mathbf{T}_{\mathbf{C}}, \mathbf{C}). \tag{9.3.1}$$

This isomorphism sends an element $T \in \mathbf{T}$ to the homomorphism

$$X \mapsto \langle v|T, X \rangle = a_1(v|T|X).$$

Since the identification $S_k = \text{Hom}_{\mathbf{C}}(\mathbf{T}_{\mathbf{C}}, \mathbf{C})$ is canonical and since the vector v is canonical, we see that the isomorphism (9.3.1) is canonical.

Recall that M_k has as basis the set of products $E_4^a E_6^b$, where $4a + 6b = k$, and S_k is the subspace of forms where the constant coefficient of their q -expansion is 0. Thus there is a basis of S_k consisting of forms whose q -expansions have coefficients in \mathbf{Q} . Let $S_k(\mathbf{Z}) = S_k \cap \mathbf{Z}[[q]]$, be the submodule of S_k generated by cusp forms with Fourier coefficients in \mathbf{Z} , and note that $S_k(\mathbf{Z}) \otimes \mathbf{Q} \cong S_k(\mathbf{Q})$. Also, the explicit formula $(\sum a_n q^n)|T_p = \sum a_{np} q^n + p^{k-1} \sum a_n q^{np}$ implies that the Hecke algebra \mathbf{T} preserves $S_k(\mathbf{Z})$.

Proposition 9.3.1. *The Fourier coefficients of each f_i are totally real algebraic integers.*

Proof. The coefficient $a_n(f_i)$ is the eigenvalue of T_n acting on f_i . As observed above, the Hecke operator T_n preserves $S_k(\mathbf{Z})$, so the matrix $[T_n]$ of T_n with respect to a basis for $S_k(\mathbf{Z})$ has integer entries. The eigenvalues of T_n are algebraic integers, since the characteristic polynomial of $[T_n]$ is monic and has integer coefficients.

The eigenvalues are real since the Hecke operators are self-adjoint with respect to the Petersson inner product. \square

Remark 9.3.2. A *CM field* is a quadratic imaginary extension of a totally real field. For example, when $n > 2$, the field $\mathbf{Q}(\zeta_n)$ is a CM field, with totally real subfield $\mathbf{Q}(\zeta_n)^+ = \mathbf{Q}(\zeta_n + 1/\zeta_n)$. More generally, one shows that the eigenvalues of any newform $f \in S_k(\Gamma_1(N))$ generate a totally real or CM field.

Proposition 9.3.3. *We have $v \in S_k(\mathbf{Z})$.*

Proof. This is because $v = \sum \text{Tr}(T_n)q^n$, and, as we observed above, there is a basis so that the matrices T_n have integer coefficients. \square

Example 9.3.4. When $k = 36$, we have

$$v = 3q + 139656q^2 - 104875308q^3 + 34841262144q^4 + 892652054010q^5 - 4786530564384q^6 + 878422149346056q^7 + \dots$$

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The normalized newforms f_1, f_2, f_3 are

$$f_i = q + aq^2 + (-1/72a^2 + 2697a + 478011548)q^3 + (a^2 - 34359738368)q^4 \\ (a^2 - 34359738368)q^4 + (-69/2a^2 + 14141780a + 1225308030462)q^5 + \dots,$$

for a each of the three roots of $X^3 - 139656X^2 - 59208339456X - 1467625047588864$.

9.3.2 General level

Now we consider the case for general level N . Recall that there are maps

$$S_k(\Gamma_1(M)) \rightarrow S_k(\Gamma_1(N)),$$

for all M dividing N and all divisor d of N/M .

The *old subspace* of $S_k(\Gamma_1(N))$ is the space generated by all images of these maps with $M|N$ but $M \neq N$. The *new subspace* is the orthogonal complement of the old subspace with respect to the Petersson inner product.

There is an algebraic definition of the new subspace. One defines trace maps

$$S_k(\Gamma_1(N)) \rightarrow S_k(\Gamma_1(M))$$

for all $M < N, M | N$ which are adjoint to the above maps (with respect to the Petersson inner product). Then f is in the new part of $S_k(\Gamma_1(N))$ if and only if f is in the kernels of all of the trace maps.

It follows from Atkin-Lehner-Li theory that the T_n acts semisimply on the new subspace $S_k(\Gamma_1(M))_{\text{new}}$ for all $M \geq 1$, since the common eigenspaces for all T_n each have dimension 1. Thus $S_k(\Gamma_1(M))_{\text{new}}$ has a basis of normalized eigenforms. We have a natural map

$$\bigoplus_{M|N} S_k(\Gamma_1(M))_{\text{new}} \hookrightarrow S_k(\Gamma_1(N)).$$

The image in $S_k(\Gamma_1(N))$ of an eigenform f for some $S_k(\Gamma_1(M))_{\text{new}}$ is called a *newform* of level $M_f = M$. Note that a newform of level less than N is not necessarily an eigenform for all of the Hecke operators acting on $S_k(\Gamma_1(N))$; in particular, it can fail to be an eigenform for the T_p , for $p | N$.

Let

$$v = \sum_f f(q^{\frac{N}{M_f}}) \in S_k(\Gamma_1(N)),$$

where the sum is taken over all newforms f of weight k and some level $M | N$. This generalizes the v constructed above when $N = 1$ and has many of the same good properties. For example, $S_k(\Gamma_1(N))$ is free of rank 1 over \mathbf{T} with basis element v . Moreover, the coefficients of v lie in \mathbf{Z} , but to show this we need to know that $S_k(\Gamma_1(N))$ has a basis whose q -expansions lie in $\mathbf{Q}[[q]]$. This is true, but we will not prove it here. One way to proceed is to use the Tate curve to construct a q -expansion map $H^0(X_1(N), \Omega_{X_1(N)/\mathbf{Q}}) \rightarrow \mathbf{Q}[[q]]$, which is compatible with the usual Fourier expansion map.³

³Where will we?

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Example 9.3.5. The space $S_2(\Gamma_1(22))$ has dimension 6. There is a single newform of level 11,

$$f = q - 2q^2 - q^3 + 2q^4 + q^5 + 2q^6 - 2q^7 + \dots$$

There are four newforms of level 22, the four $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -conjugates of

$$g = q - \zeta q^2 + (-\zeta^3 + \zeta - 1)q^3 + \zeta^2 q^4 + (2\zeta^3 - 2)q^5 \\ + (\zeta^3 - 2\zeta^2 + 2\zeta - 1)q^6 - 2\zeta^2 q^7 + \dots$$

where ζ is a primitive 10th root of unity.

Warning 9.3.6. Let $S = S_2(\Gamma_0(88))$, and let $v = \sum \text{Tr}(T_n)q^n$. Then S has dimension 9, but the Hecke span of v only has dimension 7. Thus the more “canonical looking” element $\sum \text{Tr}(T_n)q^n$ is not a generator for S .⁴

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9.4 Decomposing the anemic Hecke algebra

We first observe that it make no difference whether or not we include the Diamond bracket operators in the Hecke algebra. Then we note that the \mathbf{Q} -algebra generated by the Hecke operators of index coprime to the level is isomorphic to a product of fields corresponding to the Galois conjugacy classes of newforms.

85-1

Proposition 9.4.1. *The operators $\langle d \rangle$ on $S_k(\Gamma_1(N))$ lie in $\mathbf{Z}[\dots, T_n, \dots]$.*

Proof. It is enough to show $\langle p \rangle \in \mathbf{Z}[\dots, T_n, \dots]$ for primes p , since each $\langle d \rangle$ can be written in terms of the $\langle p \rangle$. Since $p \nmid N$, we have that⁵

$$T_{p^2} = T_p^2 - \langle p \rangle p^{k-1},$$

so $\langle p \rangle p^{k-1} = T_p^2 - T_{p^2}$. By Dirichlet's theorem on primes in arithmetic progression [Lan94, VIII.4], there is another prime q congruent to $p \pmod N$. Since p^{k-1} and q^{k-1} are relatively prime, there exist integers a and b such that $ap^{k-1} + bq^{k-1} = 1$. Then

$$\langle p \rangle = \langle p \rangle (ap^{k-1} + bq^{k-1}) = a(T_p^2 - T_{p^2}) + b(T_q^2 - T_{q^2}) \in \mathbf{Z}[\dots, T_n, \dots].$$

□

Let S be a space of cusp forms, such as $S_k(\Gamma_1(N))$ or $S_k(\Gamma_1(N), \epsilon)$. Let

$$f_1, \dots, f_d \in S$$

be representatives for the Galois conjugacy classes of newforms in S of level N_{f_i} dividing N . For each i , let $K_i = \mathbf{Q}(\dots, a_n(f_i), \dots)$ be the field generated by the Fourier coefficients of f_i .

85-7

⁴I think this because using my MAGMA program, I computed the image of v under T_1, \dots, T_{25} and the span of the image has dimension 7. For example, there is an element of S whose q -expansion has valuation 7, but no element of the Γ -span of v has q -expansion with valuation 7 or 9.

⁵See where?

85-4

Definition 9.4.2 (Anemic Hecke Algebra). The *anemic Hecke algebra* is the subalgebra

$$\mathbf{T}_0 = \mathbf{Z}[\dots, T_n, \dots : (n, N) = 1] \subset \mathbf{T}$$

of \mathbf{T} obtained by adjoining to \mathbf{Z} only those Hecke operators T_n with n relatively prime to N .

Proposition 9.4.3. *We have $\mathbf{T}_0 \otimes \mathbf{Q} \cong \prod_{i=1}^d K_i$.*

The map sends T_n to $(a_n(f_1), \dots, a_n(f_d))$. The proposition can be proved using the discussion above and Atkin-Lehner-Li theory, but we will not give a proof here.⁶

Example 9.4.4. When $S = S_2(\Gamma_1(22))$, then $\mathbf{T}_0 \otimes \mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}(\zeta_{10})$ (see Example 9.3.5). When $S = S_2(\Gamma_0(37))$, then $\mathbf{T}_0 \otimes \mathbf{Q} \cong \mathbf{Q} \times \mathbf{Q}$.

⁶Add for book.

Notes

77-1 $U^{\wedge}N_d$

77-2 Put ≥ 1 subscripts on all these

77-3 Point to same arg in sec 3

78-1 Easy warmup exer (do and add to hw)

78-2 Lazy

78-3 Lot of gaps in this claim...

78-4 Passive

78-5 Geq 1

79-1 Why?

79-2 Choppy. Ref to example

79-3 Gcd or p nmid

79-4 That, no comma. Give details or precise ref. useful for comp

79-5 Use xgcd in any way to

79-6 What if that eps thing is not 1?

79-7 Relatively?

79-9 Gcd

79-11 In the new subspace

79-12 Gcd

80-1 since eps is zero on MN divs, dont hVe to separate.

80-2 really need to talk about conductor here!

80-3 proof of this also in diamond-im was enlightening, so give it here in more generality

Notes

80-4 special case of b or c from prev section!?

80-6 (only?) also superscript would be more consistent with U notation in prev section

81-1 express on terms of alpha above

81-2 write down Up matrix on space spanned by all f_i as well. note is not semistable?

81-4 ref... and shimura

81-5 ref to earlier

$\rho_{f, \lambda} = \rho_{\lambda}$

81-7 that such a rho exists

82-1 acting on the 2d span!

82-2 exact ref

82-3 for example : give numbers... double check with sage session

82-4 explain plan of is section

82-6 all T_n !!

83-1 ref back

83-2 isom

83-3 didnt we do is already elsewhere

83-4 reference

83-5 remark that f is just the form $\sum \text{Tr}(T_n)q^n$.

83-6 so their traces are integers

84-1 same para.

84-4 always fails?

84-5 T_C not T !!

Notes

84-6 no comma, that...

85-1 make sure this galois conj class definition terminology is clear!

85-4 by Dirichlet's thm. explain. ref.

85-7 give details on sage

86-1 must have used this before?

86-2 gcd

86-4 could be good exercise

86-5 rmk that index of T_0 in T is not finite in general.