The Modular Forms Database Project

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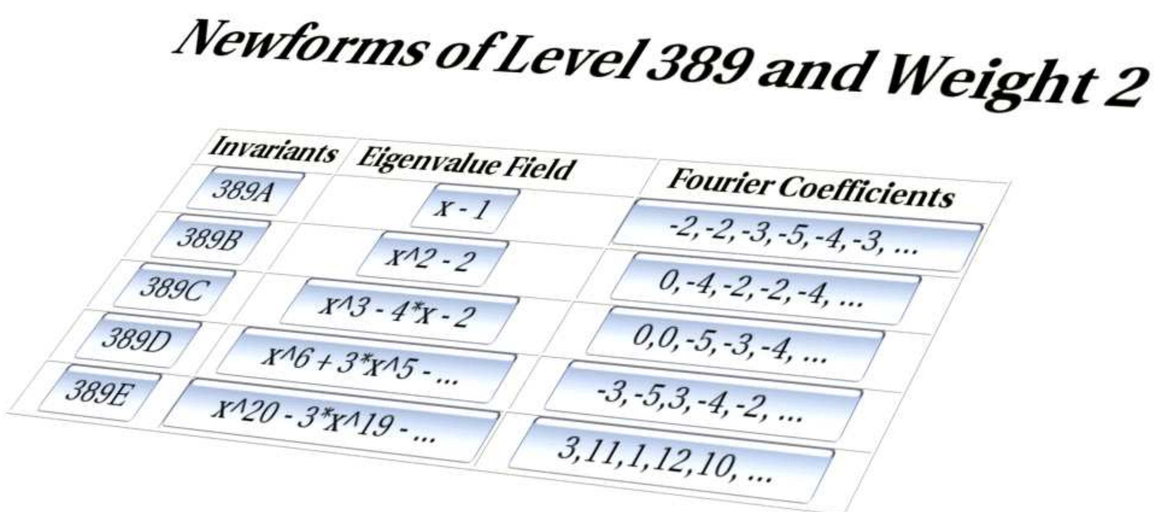
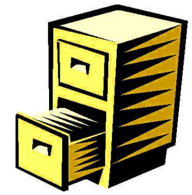


Overview of Talk

1. Computing with modular forms

2. Hardware and software

3. Live tour of the database



Goal: Large Database

Create a large database of information about modular forms for (congruence) subgroups of $\mathrm{SL}_2(\mathbf{Z})$.

- Service to the mathematical community
- Frequent thanks from people for making this data available:

```
From: Brad Emmons <braemmon@indiana.edu>
To: was@math.harvard.edu, Date: 03/03/03 07:11 pm
Dear Dr. Stein,

My name is Brad Emmons and I am a graduate student in Mathematics at
Indiana University in Bloomington.  I plan on finishing my dissertati
sometime this spring or early this summer.  My thesis deals with
finding all cases where the product of two Hecke eigenforms is anothe
Hecke eigenform, and I think you will be happy to hear that I have
used a couple of the tables that you have posted on your website to
verify some of my results.  [...]
```

Shimura

Modular Forms

Defn: A **cusp form** of integer weight k and level N is a holomorphic function

$$f : \mathfrak{h} = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\} \longrightarrow \mathbf{C}$$

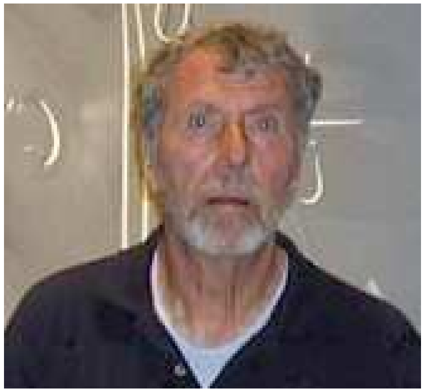
such that

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbf{Z})$ with $N \mid c$ and $a \equiv 1 \pmod{N}$, which has certain vanishing conditions at the cusps $\mathbf{Q} \cup \{\infty\}$.

$$S_k(N) = \{\text{finite dimensional space of cuspforms}\} \hookrightarrow \mathbf{C}[[q]]$$

Newforms



Oliver Atkin

Fourier Expansion:

$$f = \sum_{n=1}^{\infty} a_n q^n \quad \text{where } q(z) = e^{2\pi i z}$$

Hecke algebra:

$$\mathbf{T} = \mathbf{Z}[T_2, T_3, \ldots] \hookrightarrow \mathrm{End}(S_k(N))$$

Newform: A $\mathbf{T}$-eigenform $f = \sum a_n q^n$ normalized so that $a_1 = 1$, which does not come from level M for $M \mid N$ and $M \neq N$.

Atkin-Lehner: The structure of $S_k(N)$ as a $\mathbf{T}$-module can be completely understood in terms of newforms f.

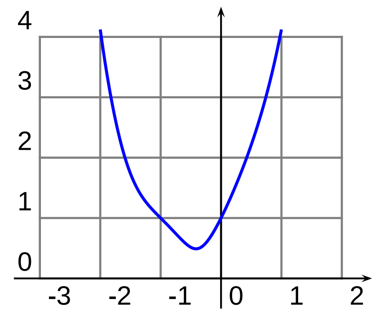
Our Job: Compute huge numbers of newforms for various N, k.

Computing Newforms

Loïc Merel

Algorithm to compute newforms in $S_k(N)$ for any N and any integer $k \geq 2$ (see Merel's LNM 1585 and Stein's Ph.D. thesis):

1. Compute **modular symbols** space $\mathcal{S}_k(N)$, which is given as $\mathbf{T}$-module by explicit generators and relations.
2. Compute $\mathcal{S}_k(N)^{\text{new}} = \ker(\mathcal{S}_k(N) \to \oplus\{\text{lower levels}\})$
3. Compute systems of T-eigenvalues $\{a_p\}$ using linear algebra tricks.



Piece of $X_1(13)(\mathbf{R})$

Example:

```
> M := ModularSymbols(Gamma1(13),2);
> S := CuspidalSubspace(M);
> Basis(S);
[
-1*{-1/2, 0} + -1*{-1/3, 0} + {-1/8, 0} + {-1/10, 0} + -1*{2/5, 1/2},
-1*{-1/2, 0} + -1/2*{-1/3, 0} + 1/2*{-1/4, 0} + 1/2*{-1/5, 0}
   + 1/2*{-1/10, 0} + -1/2*{2/5, 1/2} + -1/2*{8/17, 1/2},
-1/2*{-1/3, 0} + -1/2*{-1/4, 0} + 1/2*{-1/5, 0} + 1/2*{-1/10, 0}
   + -1/2*{2/5, 1/2} + 1/2*{8/17, 1/2},
-1*{-1/3, 0} + {-1/10, 0}
]
> CharacteristicPolynomial(HeckeOperator(S,2));
x^4 + 6*x^3 + 15*x^2 + 18*x + 9   // This is (x^2 + 3*x + 3)^2
> D := NewformDecomposition(S); D;
[
Modular symbols space of level 13, weight 2, and dimension 4 over
Rational Field (multi-character)
]
> qEigenform(D[1],5);
q + (-zeta_6 - 1)*q^2 + (2*zeta_6 - 2)*q^3 + zeta_6*q^4 + O(q^5)
```

Some Interesting Items to Compute About f

- **Invariants** of number field $K_f = \mathbf{Q}(a_1, a_2, \ldots)$ and the order $\mathcal{O}_f = \mathbf{Z}[a_1, a_2, \ldots]$ like discriminant, class number, etc. Uses Sturm bound.

- **Trace** $\mathrm{Tr}(f) = \sum \mathrm{Tr}(a_n) q^n$, simple to store and search

- **Congruences** between f and forms at other levels (Ribet's level raising and lowering) and weights (p-adic variation)

Items to Compute About Abelian Variety A_f

When $k = 2$ there is an abelian variety $A_f = J_1(N)/I_f J_1(N)$ attached to f of dimension $[\mathbf{Q}(a_2, a_3, \ldots) : \mathbf{Q}]$.

- **Equations** for A_f, especially when $\dim A_f$ small
- **Rank** of Mordell-Weil group $A_f(\mathbf{Q})$
- **Analytic Rank** $\mathrm{ord}_{s=1} L(A_f, s)$
- **Torsion** subgroup $A_f(\mathbf{Q})_{\mathrm{tor}}$ (or at least bounds on order)
- **Tamagawa** numbers of A_f
- **Canonical measure** of $A_f(\mathbf{R})$
- **Regulator** of A_f
- **Order of** $Ш(A_f)$ under Birch and Swinnerton-Dyer conjecture

Sources of Data: Elliptic Curves

John Cremona

Armand Brumer

Mark Watkins

- **Cremona's tables:** Extensive data about **all** elliptic curves of conductor ≤ 17000

- **Brumer & McGuinness:** Data about many curves of prime conductor $< 10^8$

- **Stein-Watkins:** Huge number of elliptic curves of not-too-big height (and their twists); mostly conductor $< 10^8$.

Source of Data: Modular Forms

William Stein's Watch

My MAGMA packages

- *Most non-elliptic curve data* in my database comes from this modular symbols package
- MAGMA excels at dense linear algebra over $\mathbf{Q}$, mostly because of work of Allan Steel

Technology: Software

- **MAGMA:** Enthusiastic support

- **Pari:** Generally useful

- **PostgreSQL:** Dump all data computed in huge ($>$10GB) database; possible to make queries on data or produce tables from data

- **Python:** Nice interface with database – web interface to database so anyone can access data (no online searching yet); helps to coordinate computations

- **Undergraduates:** E.g., Dimitar Jetchev has been helpful

Meccah

Technology: Hardware

- **MECCAH Cluster:** A rack of six custom-built *dual* Athlon 2000MP machines with $\geq$ 2GB memory each. (Cost: $20000 in March 2002.)

- **Sun V480:** A (new for us) quad processor 64-bit machine with 22GB (Cost: nothing – Sun education grant)

A Guided Tour

- Level 1, weight 12
- Level 37, weight 2
- Level 389, weight 2
- Running lots of jobs on MECCAH

Thank You