8.5 The Pairing Between Modular Symbols and Modular Forms

In this section we define a pairing between modular symbols and modular forms, and prove that the Hecke operators respect this pairing. We also define an involution on modular symbols, and study its relationship with the pairing. This pairing is crucial in much that follows, because it gives rise to period maps from modular symbols to certain complex vector spaces.

Fix an integer weight $k \geq 2$ and a finite-index subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$. Let $M_k(\Gamma)$ denote the space of holomorphic modular forms of weight k for Γ, and $S_k(\Gamma)$ its cuspidal subspace. Following [Mer94, §1.5], let

$$\overline{S}_k(\Gamma) = \{\overline{f} : f \in S_k(\Gamma)\}$$

denote the space of *antiholomorphic* cuspforms. Here $\overline{f}$ is the function on $\mathfrak{h}^*$ given by $\overline{f}(z) = \overline{f(z)}$.

Define a pairing

$$(S_k(\Gamma) \oplus \overline{S}_k(\Gamma)) \times \mathbb{M}_k(\Gamma) \to \mathbb{C} \tag{8.5.1}$$

by letting

$$\langle (f_1, f_2), P\{\alpha, \beta\} \rangle = \int_\alpha^\beta f_1(z) P(z,1) dz + \int_\alpha^\beta f_2(z) P(\overline{z}, 1) d\overline{z},$$

and extending linearly. Here the integral is a complex path integral along a simple path from α to β in $\mathfrak{h}$ (so, e.g., write $z(t) = x(t)+iy(t)$, where $(x(t), y(t))$ traces out the path, and consider two real integrals).

Proposition 8.5.1. *The integration pairing is well defined, i.e., if we replace $P\{\alpha, \beta\}$ by an equivalent modular symbols (equivalent modulo the left action of Γ), then the integral is the same.*

Proof. We use the change of variables formulas for integration and the fact that $f_1 \in S_k(\Gamma)$ and $f_2 \in \overline{S}_k(\Gamma)$. For example, if $k = 2$, $g \in \Gamma$ and $f \in S_k(\Gamma)$, then

$$\begin{aligned}
\langle f, g\{\alpha, \beta\} \rangle &= \langle f, \{g(\alpha), g(\beta)\} \rangle \\
&= \int_{g(\alpha)}^{g(\beta)} f(z) dz \\
&= \int_\alpha^\beta f(g(z)) dg(z) \\
&= \int_\alpha^\beta f(z) dz = \langle f, \{\alpha, \beta\} \rangle,
\end{aligned}$$

where $f(g(z))dg(z) = f(z)dz$ because f is a weight 2 modular form. For the case of arbitrary weight $k > 2$, see Exercise 8.4 □

The integration pairing is very relevant to the study of special values of L-functions. The L-function of a cusp form $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ is

$$L(f,s) = (2\pi)^s \Gamma(s)^{-1} \int_0^\infty f(it) t^s \frac{dt}{t} \qquad (8.5.2)$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad (8.5.3)$$

The equality of the integral and the Dirichlet series follows by switching the order of summation and integration, which is justified using standard estimates on $|a_n|$ (see, e.g., [Kna92, §VIII.5]).

For each integer j with $1 \leq j \leq k-1$, we have setting $s = j$ and making the change of variables $t \mapsto -it$ in (8.5.2), that

$$L(f,j) = \frac{(-2\pi i)^j}{(j-1)!} \cdot \left\langle f,\ X^{j-1} Y^{k-2-(j-1)} \{0, \infty\} \right\rangle . \qquad (8.5.4)$$

The integers j as above are called *critical integers*, and when f is an eigenform, they have deep conjectural significance. We will discuss explicit computation of $L(f,j)$ in Chapter 10.

Theorem 8.5.2 (Shokoruv)**.** *The pairing $\langle \cdot\,,\, \cdot \rangle$ is nondegenerate when restricted to cuspidal modular symbols:*

$$\langle \cdot\,,\, \cdot \rangle : (S_k(\Gamma) \oplus \overline{S}_k(\Gamma)) \times \mathbb{S}_k(\Gamma) \to \mathbb{C}.$$

The pairing is also compatible with Hecke operators. Before proving this, we define an action of *Hecke operators* on $M_k(\Gamma_1(N))$ and on $\overline{S}_k(\Gamma_1(N))$. The definition is very similar to the one we gave in Section 2.4 for modular forms of level 1. For a positive integer n, let R_n be a set of coset representatives for $\Gamma_1(N)\backslash\Delta_n$ from Lemma 8.3.1. For any $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$ and $f \in M_k(\Gamma_1(N))$ set

$$f|[\gamma]_k = \det(\gamma)^{k-1}(cz+d)^{-k} f(\gamma(z)).$$

Also, for $f \in \overline{S}_k(\Gamma_1(N))$, set

$$f|[\gamma]_k' = \det(\gamma)^{k-1}(c\overline{z}+d)^{-k} f(\gamma(z)).$$

Then for $f \in M_k(\Gamma_1(N))$,

$$T_n(f) = \sum_{\gamma \in R_n} f|[\gamma]_k$$

and for $f \in \overline{S}_k(\Gamma_1(N))$,

$$T_n(f) = \sum_{\gamma \in R_n} f|[\gamma]_k'.$$

This agrees with the definition from 2.4 when $N = 1$.

Remark 8.5.3. If Γ is an arbitrary finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$, then we can define operators T_Δ on $M_k(\Gamma)$ for any Δ with $\Delta\Gamma = \Gamma\Delta = \Delta$ and $\Gamma\backslash\Delta$ finite. For concreteness we do not do the general case here or in the theorem below, but the proof is exactly the same (see [Mer94, §1.5]).

Finally we prove the promised Hecke compatibility of the pairing. This proof should convince you that the definition of modular symbols is sensible, in that they are natural objects to integrate against modular forms.

Theorem 8.5.4. *If* $f = (f_1, f_2) \in S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))$ *and* $x \in \mathbb{M}_k(\Gamma_1(N))$, *then for any* n,

$$\langle T_n(f), x\rangle = \langle f, T_n(x)\rangle.$$

Proof. We follow [Mer94, §2.1] (but with more details), and will only prove the theorem when $f = f_1 \in S_k(\Gamma_1(N))$, the proof in the general case being the same.

Let $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$, $P \in \mathbb{Z}_{k-2}[X, Y]$, and for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Q})$, set $j(g, z) = cz + d$. Let n be any positive integer, and let R_n be a set of coset representatives for $\Gamma_1(N)\backslash\Delta_n$ from Lemma 8.3.1.

We have

$$\begin{aligned}\langle T_n(f), P\{\alpha, \beta\}\rangle &= \int_\alpha^\beta T_n(f)P(z,1)dz \\ &= \sum_{\delta\in R}\int_\alpha^\beta \det(\delta)^{k-1} f(\delta(z))j(\delta, z)^{-k}P(z,1)dz.\end{aligned}$$

Now for each summand corresponding to the $\delta \in R$, make the change of variables $u = \delta z$. Thus we make $\#R$ change of variables. Also, we will use the notation $\tilde{g} = \left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right) = \det(g)\cdot g^{-1}$ for $g \in \mathrm{GL}_2(\mathbb{Q})$. We have

$$\langle T_n(f), P\{\alpha, \beta\}\rangle = \sum_{\delta\in R}\int_{\delta(\alpha)}^{\delta(\beta)} \det(\delta)^{k-1} f(u)j(\delta, \delta^{-1}(u))^{-k}P(\delta^{-1}(u), 1)d(\delta^{-1}(u))$$

Note that $\delta^{-1}(u) = \tilde{\delta}(u)$, since a linear fractional transformation is unchanged by a nonzero rescaling of a matrix that induces it. Thus by the quotient rule, using that $\tilde{\delta}$ has determinant $\det(\delta)$, we see that

$$d(\delta^{-1}(u)) = d(\tilde{\delta}(u)) = \frac{\det(\delta)du}{j(\tilde{\delta}, u)^2}.$$

We next show that

$$j(\delta, \delta^{-1}(u))^{-k}P(\delta^{-1}(u), 1) = j(\tilde{\delta}, u)^k \det(\delta)^{-k} P(\tilde{\delta}(u), 1). \tag{8.5.5}$$

From the definitions, and again using that $\delta^{-1}(u) = \tilde{\delta}(u)$, we see that

$$j(\delta, \delta^{-1}(u)) = \frac{\det(\delta)}{j(\tilde{\delta}, u)},$$

which proves that (8.5.5) holds. Thus

$$\langle T_n(f), P\{\alpha,\beta\}\rangle = \sum_{\delta\in R}\int_{\delta(\alpha)}^{\delta(\beta)} \det(\delta)^{k-1} f(u) j(\tilde{\delta},u)^k \det(\delta)^{-k} P(\tilde{\delta}(u),1)\frac{\det(\delta)du}{j(\tilde{\delta},u)^2}$$

Next we use that

$$(\delta.P)(u,1) = j(\tilde{\delta},u)^{k-2}P(\tilde{\delta}(u),1).$$

To see this, note that $P(X,Y) = P(X/Y,1)\cdot Y^{k-2}$. Using this we see that

$$\begin{aligned}(\delta.P)(X,Y) &= (P\circ\tilde{\delta})(X,Y)\\ &= P\left(\tilde{\delta}\left(\frac{X}{Y}\right),1\right)\cdot\left(-c\cdot\frac{X}{Y}+a\right)^{k-2}\cdot Y^{k-2}.\end{aligned}$$

Now substituting $(u,1)$ for $(X,1)$, we see that

$$(\delta.P)(u,1) = P(\tilde{\delta}(u),1)\cdot(-cu+a)^{k-2},$$

as required. Thus finally

$$\begin{aligned}\langle T_n(f), P\{\alpha,\beta\}\rangle &= \sum_{\delta\in R}\int_{\delta(\alpha)}^{\delta(\beta)} f(u)j(\tilde{\delta},u)^{k-2}P(\tilde{\delta}(u),1)du\\ &= \sum_{\delta\in R}\int_{\delta(\alpha)}^{\delta(\beta)} f(u)\cdot((\delta.P)(u,1))du\\ &= \langle f, T_n(P\{\alpha,\beta\})\rangle.\end{aligned}$$

□

8.6 Exercises

8.1 Suppose M is an integer multiple of N. Prove that the natural map $(\mathbb{Z}/M\mathbb{Z})^* \to (\mathbb{Z}/N\mathbb{Z})^*$ is surjective.

8.2 Prove that $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/N\mathbb{Z})$ is surjective (see Lemma 8.4.6).

8.3 Compute $\mathbb{M}_3(\Gamma_1(3))$ explicitly. List each Manin symbol, the relations they satisfy, compute the quotient, etc. Find the matrix of T_2. (Check: The dimension of $\mathbb{M}_3(\Gamma_1(3))$ is 2, and the characteristic polynomial of T_2 is $(x-3)(x+3)$.)

8.4 Finish the proof of Proposition 8.5.1.