Math 129: Algebraic Number Theory
Homework Assignment 3

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Due: Thursday, March 4, 2004

The problems:

1. (a) Suppose K is a number field of degree 2. Prove that $\mathcal{O}_K = \mathbf{Z}[a]$ for some $a \in \mathcal{O}_K$.
   (b) Prove that if K and K' are two number fields of degree 2 and $\mathrm{Disc}(\mathcal{O}_K) = \mathrm{Disc}(\mathcal{O}_{K'})$ then $K = K'$.

2. (*) Does there exist a number field K of degree 4 such that $\mathcal{O}_K \neq \mathbf{Z}[a]$ for all $a \in \mathcal{O}_K$? If so, give an explicit example.

3. Let K be the quintic number field generated by a root of $x^5+7x^4+3x^2-x+1$. Draw a diagram (be creative) that illustrates the factorization of every prime $p \in \mathbf{Z}$, with $p < 100$, in $\mathcal{O}_K$.

4. (Problem 1.9 in Swinnerton-Dyer) Show that the only solutions $x, y \in \mathbf{Z}$ to $y^2 = x^3 - 13$ are given by $x = 17, y = \pm 70$, as follows. Factor the equation $y^2 + 13 = x^3$ in the number field $\mathbf{Q}(\sqrt{-13})$, which has class number 2. Show that if x, y is an integer solution then the ideal $(y + \sqrt{-13})$ must be the cube of an ideal, and hence $y + \sqrt{-13} = (a + b\sqrt{-13})^3$; thus $1 = b(3a^2 - 13b^2)$.

5. Suppose I and J are ideals in the ring $\mathcal{O}_K$ of integers of a number field K. Does $IJ = I \cap J$? Prove or give a counterexample.

6. Let $\mathcal{O}_K$ be the ring of integers $\mathbf{Q}(\sqrt{5})$, and let

   $$I = (5, 2+\sqrt{5}) \qquad \text{and} \qquad J = (209, (389+\sqrt{5})/2)$$

   be integral ideals of $\mathcal{O}_K$.

   (a) Find an element of $\mathcal{O}_K$ that is congruent to $\sqrt{5}$ modulo I and is congruent to $1 - \sqrt{5}$ modulo J.
   (b) What is the cardinality of $(\mathcal{O}_K/I) \oplus (\mathcal{O}_K/J)$?
   (c) Find an element $a \in I$ such that $(a)/I$ is coprime to J.

7. Let $\mathcal{O}_K$ be the ring of integers of a number field K, and suppose K has exactly $2s$ complex embeddings. Prove that the sign of $\mathrm{Disc}(\mathcal{O}_K)$ is $(-1)^s$.