Sujatha: Intro to Hida Theory

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$\mathbb{Q}$; $G_{\mathbb{Q}} = Gal(\overline{\mathbb{Q}}/\mathbb{Q})$

Question: Understand $G_{\mathbb{Q}}$; partially achieved by CFT. Representation.

Try to understand $G_{\mathbb{Q}}$ through "Galois representations"

Ex A: $E/_{\mathbb{Q}}$ elliptic curve, p odd prime

fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$, $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$

$E_{p^n} = E_{p^n}(\overline{\mathbb{Q}})$ p^n-torsion.

$E_{p^\infty} = \bigcup_{n \geq 0} E_{p^n}(\overline{\mathbb{Q}})$

$E_{p^\infty} \simeq \left(\mathbb{Q}_p/\mathbb{Z}_p \oplus \mathbb{Q}_p/\mathbb{Z}_p\right) \circlearrowleft G_{\mathbb{Q}}$

$T_p(E) = \varprojlim E_{p^n}$; $(\mathbb{Z}_p{}^2) \circlearrowleft G_{\mathbb{Q}}$

$T_p(E) \otimes \mathbb{Q}_p = V_p(E)$, $\rho_{E,p}: G_{\mathbb{Q}} \to GL_2(\mathbb{Q}_p)$.

Assume E is ordinary at p.

$\Rightarrow V = V_p\hat{E} \oplus V_p\tilde{E}$ as a repn over $G_p = Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$

(i) $\rho_E|_{G_p} = \begin{pmatrix} * & * \\ 0 & \psi \end{pmatrix}$, where ψ is unramified $(\psi|_{I_p} = 1)$

(ii) ρ_E unramified outside finite set of places, e.g., all $\ell \nmid N_E p$.
$\ell \nmid N_p$

(iii) $Tr(\rho_E(Frob_\ell)) = a_\ell$

$E \leftrightarrow f_E = \sum a_n q^n$

Ex B: $f = \sum a_n q^n$ newform, weight k, tame level N; then $K_f = \mathbb{Q}(\ldots a_n \ldots)$. Assume further that $p \nmid a_p$ (ordinary),

Then $\exists$ a Galois repn $\rho_f: G_{\mathbb{Q}} \to GL(V_f)$,

V_f = 2 dim'l vector space over $K_{f,\mathfrak{p}}$ for $\mathfrak{p}|p$.

(i) ρ_f unramified outside finite set: $\ell \nmid N_p$.

(ii) $\rho_f|_{G_{\mathbb{Q}_p}}$ is upper triangular $\begin{pmatrix} * & * \\ 0 & \psi \end{pmatrix}$

(iii) $Tr(\rho_f(Frob_\ell)) = a_\ell$ $(\ell \nmid N_p)$.

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§ Hida theory simplified. Hida's insight = one can package these repn's as "UNITS" of a "BIG" representation.

Let $\mathcal{R} = \mathbb{Z}_p[[w]]$; $\mathcal{F} = Frac(\mathcal{R})$

Consider a formal q-expansion

$$F = \sum_{0 \leq n < \infty} \mathcal{A}(n,F) q^n \quad \text{where } \mathcal{A}(n,F) \in \overline{\mathcal{F}} = \text{fixed algebraic closure of } \mathcal{F}.$$

- Assume further that the ring generated over $\mathcal{R}$ by the $\mathcal{A}(n,F)$ is just $\mathcal{R}$, i.e. all $\mathcal{A}(n,F) \in \mathcal{R}$.
- Let $\mathfrak{p}_k \subseteq \mathcal{R}$ be prime ideal given by

$$((1+w) - (1+p)^k), \quad k \geq 2 \text{ integer}.$$

$$\varphi_k : \mathcal{R} \longrightarrow \mathcal{R}/\mathfrak{p}_k = \mathcal{O}_k$$

and extend to series.

$\mathcal{O}_k$
$| < \infty$
$\mathbb{Z}_p$

Let $\varphi_k(F) = F_{/k} = f_k$
$= \sum \mathcal{A}(n,k) q^n \in \mathcal{O}_k[[q]]$, of tame level N and character ψ

Defn: F is an ordinary, cuspidal, $\mathcal{R}$-adic form if f_k is a modular form, ordinary cuspidal eigenform, level Np^r, character $\psi\chi^{2-k}$, weight $k-2$.

Hida's Theorem: (i) Let F be an $\mathcal{R}$-adic ordinary cuspidal eigenform, tame level N. Then $\exists$ a continuous irreducible representation $\rho_F : G_{\mathbb{Q}} \to GL_2(\mathcal{F})$, such that

(a) ρ_F is unramified outside Np

(b) If $\ell \nmid Np$ then $tr(\rho_F(Frob_\ell)) = \mathcal{A}(\ell, F)$.

(c) $\rho_F|_{\mathbb{Q}_p} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ unramified.

(Sujatha continued)

(ii) Any $\underline{p\text{-stabilized}}$ newform f_k of weight k and tame level N occurs as "the k-th member" of an Λ-adic form F.

F = a Hida family.

__Fact:__ All residual representations of the members in a Hida family are __isomorphic__, at least for $\boxed{\text{congruent weights}}$:

$$\rho_k : G_{\mathbb{Q}} \to GL_2(\mathcal{O}_\pi) \to GL_2(\mathbb{F}_r)$$

Suppose E_1, E_2 elliptic curves $/\mathbb{Q}$

p = prime ordinary for both curves

Assume $E_1[p] \simeq E_2[p]$, absolutely irred, as $G_{\mathbb{Q}}$-modules.

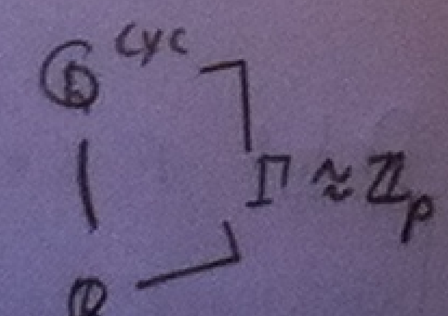
__Greenberg-Vatsal:__

For an "ordinary" Galois repn $\rho : G_{\mathbb{Q}} \to GL(V)$ can define the Selmer group, $\mathrm{Sel}(\rho)$, e.g., $\rho = \rho_E$,

$\mathrm{Sel}(\rho) = \mathrm{Sel}(E/\mathbb{Q})$ has the following property

$$0 \to E(\mathbb{Q}) \otimes \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}(E/\mathbb{Q}) \to Ш(E/\mathbb{Q})[p^\infty] \to 0$$

$$\mathrm{Sel}(E/\mathbb{Q}) \subseteq H^1(\mathbb{Q}, E_{p^\infty})$$



$$\underset{\circlearrowleft \Gamma}{\mathrm{Sel}(E/\mathbb{Q}^{cyc})} \subseteq H^1(\mathbb{Q}^{cyc}, E_{p^\infty}).$$

Let $X(E/\mathbb{Q}^{cyc})$ = pontryagin dual = $\mathrm{Hom}(\mathrm{Sel}(E/\mathbb{Q}^{cyc}), \mathbb{Q}_p/\mathbb{Z}_p)$

- Modulo certain conjectures:

$$X(E/\mathbb{Q}^{cyc}) \begin{cases} X(E/\mathbb{Q}^{cyc})(p) \\ Y(E/\mathbb{Q}^{cyc}) \overset{\text{def}}{=} \dfrac{X(E/\mathbb{Q}^{cyc})}{X(E/\mathbb{Q}^{cyc})(p)} \end{cases}$$

__Believe:__ Up to finite modules, the whole structure is captured by two invariants: μ-invariant, λ-invariant

$$X(E/\mathbb{Q}^{cyc})[p^k] = X(E/\mathbb{Q}^{cyc})[p^{k+1}] \text{ for } k \gg 0$$

__μ-invariant:__ $X(E/\mathbb{Q}^{cyc})[p^{k+1}] \big/ X(E/\mathbb{Q}^{cyc})[p^k]$ is a module over $\mathbb{F}_p[[\Gamma]]$

$$\mathrm{rank}_{\mathbb{F}_p[[\Gamma]]} \sum_k \frac{X(E/\mathbb{Q}^{cyc})[p^{k+1}]}{X(E/\mathbb{Q}^{cyc})[p^k]} \overset{\text{def}}{=} \mu\text{-invariant}.$$

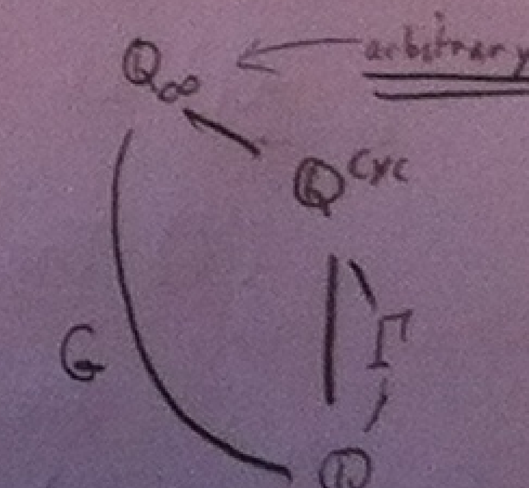
__Equivalent:__ $X(E/\mathbb{Q}^{cyc}) \underset{\text{finite}}{\overset{\text{up to}}{\sim}} \bigoplus \dfrac{\mathbb{Z}_p[[T]]}{p^{k_i}} \oplus_j \dfrac{\mathbb{Z}_p[[T]]}{f_j(T)}$

__μ-invariant__ $= \sum k_i$

__λ-invariant__ $= \sum \deg(f_j) = \mathbb{Z}_p$-rank of $Y(E/\mathbb{Q}^{cyc})$

__Theorem (Emerton-Weston-Pollack):__ Under certain conditions, $\mu = 0$ for one member of a Hida family $\Rightarrow \mu = 0$ for all other members; further the λ-invariants are equal.

__Generalize to noncommutative setting:__



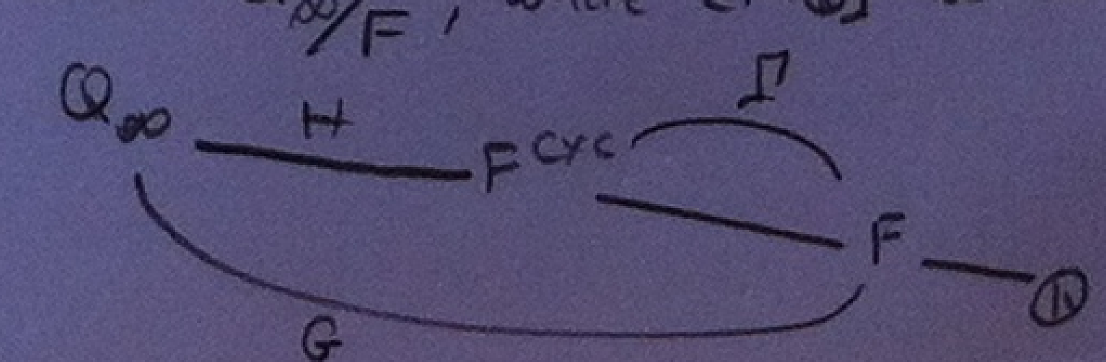
$$X(E/\mathbb{Q}_\infty) \subseteq H^1(\overline{\mathbb{Q}}/\mathbb{Q}_\infty, E)$$

$G \circlearrowleft$ ⟶ module over $\mathbb{Z}_p[[G]] = \varprojlim_{G' \triangleleft G} \mathbb{Z}_p[[G/G']]$

$$X(E/\mathbb{Q}_\infty) \begin{cases} X(E/\mathbb{Q}_\infty)(p) \leftarrow \text{can define } \mu\text{-invariant, uses } \mathrm{rk}_{\mathbb{F}_p[[G]]} \\ \boxed{Y(E/\mathbb{Q}_\infty)(p) = X(E/\mathbb{Q}_\infty) \big/ X(E/\mathbb{Q}_\infty)(p)} \end{cases}$$

$$\mu\text{-invariant} = \sum_{i=0}^{?} \mathrm{rk}_{\mathbb{F}_p[[G]]} \frac{X[p^{i+1}]}{X[p^i]}$$

Work with extensions $\mathbb{Q}_\infty/F$, where $[F:\mathbb{Q}] < \infty$ s.t. $\mathbb{Q}_\infty \supseteq F^{cyc}$.



$$\mathbb{Z}_p[[H]] \subseteq \mathbb{Z}_p[[G]]$$

$Y(E/\mathbb{Q}_\infty)(p)$ is f.g. as a module over $\mathbb{Z}_p[[H]]$.

- Assume G is a p-adic Lie group with no elts of order p.

- <u>Ex</u>:

$$\begin{tikzcd}
 & & F^{cyc}(m^{1/p^\infty}) \\
 & F^{cyc} \arrow[ur, dash] & \\
F = \mathbb{Q}(\mu_p) \arrow[ur, dash] \arrow[uurr, dash, bend left, "\mathbb{Z}_p \rtimes \mathbb{Z}_p"] & & \\
\mathbb{Q} \arrow[u, dash] & &
\end{tikzcd}$$

$m \in \mathbb{Z}$ not div by a p.

<u>λ-invariant</u> $= \operatorname{rank}_{\mathbb{Z}_p[[H]]} (Y(E/\mathbb{Q}_\infty))$

<u>Theorem</u>: "If $\mu = 0$ for one weight, then $\mu = 0$ for all"

(a) Let $\mathbf{F}$ be an R-adic ordinary cuspidal eigenform, and let $f_k = \mathbf{F}/_k$. Consider the dual Selmer group $X(F/_k/\mathbb{Q}_\infty)$.

(i) If $\mu(X(F/_k)/\mathbb{Q}_\infty) = 0$ for some weight $k \geq 2$, then $\mu = 0$ $\forall$ weights

(ii) The λ-invariants ($= \mathbb{Z}_p[[H]]$-rank of $Y(F/_k/\mathbb{Q}_\infty)$) are all equal.

<u>Ex</u>: $E = X_0(11)$, $p = 11$, F_E = Hida family with $F_{E,2} = f_E$.

<u>$k=12$</u>: $f_{12}^* = 11$-stabilized Ramanujan form Δ but <u>level 11</u>.

$\mu_E = 0$, $\lambda_E = 1$, so get something about Δ.