Analytically Bounding Ranks of Elliptic Curves
(assuming RH+BSD) — Jon Bober

Quick Idea: 1. BSD: $\text{rank}(E) = \underset{s=\frac{1}{2}}{\text{ord}}\, L(E,s)$

2. Trivial observation: If $f(x) \geq 0$, then $\frac{1}{f(0)} \sum_{\substack{L(\frac{1}{2}+i\gamma)=0 \\ \gamma\in\mathbb{R}}} f(\gamma) \geq \text{rank}(E)$
$f(0)>0$
with multiplicities

3. For some f, can calculate sum quickly.

Ranks of elliptic curves:

$E/_{\mathbb{Q}}$ $\quad E(\mathbb{Q}) \approx \mathbb{Z}^r \times E(\mathbb{Q})_{tor}$ (Theorem of Mordell) ..
$r = \text{rank}(E)$

Open Questions:

1. What values does r take? (Any nonnegative integer?)
2. How often does $r=n$, $n\geq 0$?
(50% $r=0$, 50% $r=1$, ∞'ly often anything else)

(Bhargava — Shakar)
$r=0$ for a positive proportion of curves, ordered by "naive height"
$\rightarrow$ Average rank <1. [Ralph: what about lower bound on average rank?]

3. Is there algorithm to compute r?
Yes, if $\#Ш < \infty$.
$0 \to E(\mathbb{Q})/_{nE(\mathbb{Q})} \to Sel^{(n)}(E) \to Ш[n] \to 0$
In principle, $Sel^{(n)}(E)$ can be computed. Some n with $Ш[n]=0$. Etc.]
In practice, $Sel^{(2)}(E)$ (Sage, mwrank)
$Sel^{(n)}(E)$, $n=2,3,4.$ Stoll, Cremona, etc in Magma.

Also: Easy to write down a curve E where E.selmer_rank() takes "forever".

Can also try to use L-functions

BSD: $\text{rank}(E) = \underset{s=1/2}{\text{ord}}\, L(E,s)$

So: Compute $L(E,s), L'(E,s), \ldots, L^{(n)}(E,s)$
$\rightsquigarrow$ Can get upper bound (assuming BSD)

Problem: No analytic way to tell diff between r zeros at $\frac{1}{2}$ and r zeros near $\frac{1}{2}$.

Also: Complexity of computing $L^{(n)}(\frac{1}{2},E)$ is $O(\sqrt{N_E})$.
If $N_E \sim 10^{150}$, this is hopeless.

L-functions

$$L(E,s) = \sum \frac{a_n}{n^s} = \prod_p L_p(p^{-s})^{-1}$$

Where $a_p = \frac{p+1-\#E(\mathbb{Z}/p\mathbb{Z})}{\sqrt{p}}$

Log-derivative:

$$-\frac{L'}{L}(E,s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}$$

$L_p(p^{-s}) = (1-\alpha_p p^{-s})(1-\beta_p p^{-s})$; $\alpha_p = 0$ if p additive (not mult), $\beta_p = 0$ if p bad; $|\alpha_p| = |\beta_p| = 1$, if p good

log, taylor, diff, simplify, and get $c(n) = \begin{cases} (\alpha(p)^k + \beta(p)^k)\log(p) & \text{if } n=p^k, k>0 \\ 0 & \text{otherwise.} \end{cases}$

Tool to evaluate sum from before:

Explicit formula:

If $f(z)$ is "nice" (e.g. entire) and $f(x)\in\mathbb{R}$ for $x\in\mathbb{R}$.

$$\sum_{\substack{L(\frac{1}{2}+i\gamma)=0 \\ |\text{Im}(\gamma)|<1}} f(\gamma) = \hat{f}(0)\frac{\log N}{2\pi} - \hat{f}(0)\frac{\log 2\pi}{\pi} + \frac{1}{\pi}\text{Re}\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1+it) f(t)\,dt - \frac{1}{2\pi}\sum_{n=1}^{\infty} \frac{c(n)}{n^{1/2}} \left(\hat{f}\left(\frac{\log n}{2\pi}\right) + \hat{f}\left(-\frac{\log n}{2\pi}\right) \right)$$

$$\boxed{\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t}\, dx}$$

$$= \int F(s) \frac{L'(E,s)}{L(E,s)}\, ds$$

comes out of standard complex analysis crap.

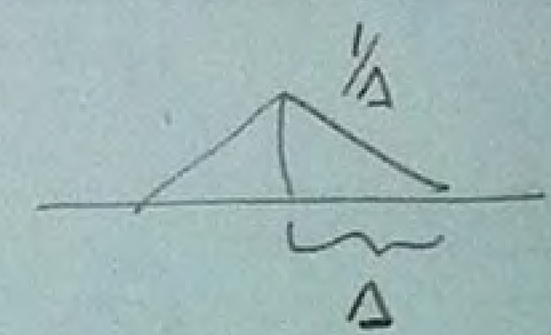
Some nice choices of F:

$$F(x) = e^{-\pi i x^2} \longleftrightarrow \hat{f}(t) = e^{-\pi t^2}$$

Anything where $\hat{f}(t) = 0$ for $|t| < \log 2 / 2\pi$

Or

$$f(x;\Delta) = \left(\frac{\sin \Delta\pi x}{\Delta \pi x}\right)^2$$



$$\longleftrightarrow \hat{f}(t) = \begin{cases} \frac{1}{\Delta}\left(1 - \frac{|t|}{\Delta}\right) & \text{for } |t| < \Delta \\ 0 & |t| \geq \Delta \end{cases}$$

With this f, explicit formula becomes:

have to overcome this

$$\text{rank}(E) \leq \sum f(\gamma;\Delta) = \boxed{\frac{\log N}{2\pi\Delta}} - \frac{\log(\pi)}{\pi\Delta} + \frac{1}{\pi}\text{Re}\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1+it) f(t)\,dt$$

$$- \frac{1}{\Delta\pi}\underbrace{\sum_p \log p \sum_{k=1}^{\infty} \frac{\alpha(p)^k + \beta(p)^k}{p^{k/2}}\left(1 - \frac{k\log p}{2\pi\Delta}\right)}_{\text{finite sum } p^k \leq \exp(2\pi\Delta)}$$

assuming RH+BSD

Applications:

Mestre: F(x Δ; bound $|\alpha(p)^k + \beta(p)^k| < 2$ → get A&B with

rank(E) ≤ A $\log N_E$ + B for all E.

→ Take $\Delta = \frac{\log\log N}{2\pi}$ and get $\text{rank}(E) \ll \frac{\log N_E}{\log\log N_E}$

(Bober):

Can also put in specific curves and see what comes out.

Examples from Dujella's website:

E_{20} E_{21} E_{22} E_{23} E_{24} E_{28}

(Elkies 2006)

Root numbers all match up with parity of rank.

cond $(E_{28}) \approx 3.5 \times 10^{141}$

(Largest known exact rank is 19.)

import psage.ellcurve.xxx.rankbound

Curve	$\sim \log N_E$	Δ	$\sum f(\gamma)$
E_{20}	170	2	21.70
E_{21}	196.68	2.5	22.68
E_{22}	182.72	2.0	23.71
E_{23}	205	2.5	24.49
E_{24}	219.93	2.5	25.57
E_{28}	325.90	3.2	31.3

really big

6 minutes

4.0 ~ few days.

Theorem (Bober): BSD + RH ⟹ E_n has rank n for $n = 20, 21, 22, 23, 24$

E_{28} has rank 28 or 30.

With this F:
$\leq \sum_{\gamma} F(\gamma;\Delta) = \boxed{\frac{\log N}{2\pi\Delta}} - \frac{\log \pi}{\pi\Delta} + \frac{1}{\pi} \mathrm{Re} \int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}(1+it)F(t)\,dt$
$- \frac{1}{\Delta\pi} \underbrace{\sum_{p} \log p \sum_{k=1} \frac{\alpha(p)^k + \beta(p)^k}{p^{k/2}}}_{p^k < \exp(2\pi\Delta)}$