Toward a Generalization of the Gross-Zagier Conjecture: II

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December 2009

Every time I have talked about generalizing Gross-Zagier, people (e.g., Mazur, Rubin, Dasgupta, etc.) have suggested I take exterior powers. This 2-page note is about one perspective on how to do so, which interestingly leads to some possibly new questions that may be possible to answer.

1 Conjecture

Let E be an optimal elliptic curve over $\mathbb{Q}$ of analytic rank $r_{\mathrm{an}}(E/\mathbb{Q}) \geq 1$. Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic imaginary field with discriminant $D \leq -5$ that satisfies the Heegner hypothesis for E such that $r_{\mathrm{an}}(E^D) \leq 1$.

Do not fix a prime number p. For each prime number ℓ that is inert in K, we define a finite index subgroup $W_\ell \subset E(K)$ as follows. Let $M = \gcd(a_\ell, \ell+1)$, where for any p prime, $a_p = a_p(E)$ is the trace of Frob_p on E. Let λ be a square-free product of either $r_{\mathrm{an}}(E/\mathbb{Q}) - 1$ or $r_{\mathrm{an}}(E/\mathbb{Q})$ inert primes p_i such that $M \mid \gcd(a_{p_i}, p_i + 1)$ for each i. Then reducing the Kolyvagin point $P_\lambda \in E(K[\lambda])$ modulo any prime over ℓ of the ring class field $K[\lambda]$ yields a well defined point

$$\overline{P}_\lambda \in E(\mathbb{F}_{\ell^2}) \otimes (\mathbb{Z}/M\mathbb{Z}).$$

This point $\overline{P}_\lambda$ is well-defined because changing the choice of prime is the same as applying an automorphism in $G = \mathrm{Gal}(K_\lambda/K)$, and our hypothesis on λ implies that

$$[P_\lambda] \in (E(K_\lambda) \otimes (\mathbb{Z}/M\mathbb{Z}))^G.$$

There is a natural reduction map $E(K) \to E(\mathbb{F}_{\ell^2}) \otimes (\mathbb{Z}/M\mathbb{Z})$, and we let W_ℓ be the inverse image in $E(K)$ of the subgroup of $E(\mathbb{F}_{\ell^2}) \otimes (\mathbb{Z}/M\mathbb{Z})$ generated by all $\overline{P}_\lambda$.

The definition of W_ℓ depends on a choice of prime ℓ. The subgroup

$$\bigcap_{\text{inert } \ell} W_\ell \subset E(K)$$

is canonical, but it turns out that it does not satisfy a Gross-Zagier style formula in general; indeed, away from non-surjective primes and 2, we expect that the real part of the above subgroup should just equal $IE(\mathbb{Q})$ for $I = \sqrt{\#\text{Ш}(E/K)} \cdot \prod c_p$.

Let $t = r_{\mathrm{an}}(E/K)$. Instead, we consider the subgroup

$$V = \bigcap_{\text{inert } \ell} \left(\bigwedge^t W_\ell \right) \subset \bigwedge^t E(K).$$

Proposition 1.1. *If V has positive rank, then Kolyvagin's conjecture that $0 \neq \{\tau\} \subset \mathrm{H}^1(K, E[p^\infty])$ is true for every odd prime p such that $\overline{\rho}_{E,p}$ is surjective.*

Proposition 1.2. *If V has positive rank, then $(\bigwedge^t E(K))_{/\,\mathrm{tor}}$ has rank 1.*

Proof. Since V has positive rank, Kolyvagin's conjecture is true, so Kolvyagin's structure theorem implies that $E(K)$ has rank *at most* t. Thus $(\bigwedge^t E(K))_{/\,\mathrm{tor}}$ has rank at most 1. Since V is a subgroup of positive rank, the rank of $(\bigwedge^t E(K))_{/\,\mathrm{tor}}$ is also at least 1. $\square$

If V has positive rank, we define a height function on $\bigwedge^t E(K)$ such that the height of $x = x_1 \wedge \cdots \wedge x_t \in E(K)$ is the regulator of the subgroup of $E(K)$ generated by $x_1, \ldots, x_t$. Then $\mathrm{Reg}(V) = I^2 \,\mathrm{Reg}(E(K))$, where

$$I = \left[(\bigwedge^t E(K))_{/\,\mathrm{tor}} \,:\, V_{/\,\mathrm{tor}} \right].$$

If V is torsion then we define $\mathrm{Reg}(V) = 0$.

Let $L^{(*)}(E/K, 1)$ be the leading coefficient of the expansion about 1 of $L(E/K, s)$.

Conjecture 1.3 (Stein)**.** *We have*

$$\frac{L^{(*)}(E/K,1)}{\Omega_{E/K}} = \mathrm{Reg}(V).$$

The above statement when $t = 1$ is just the classical Gross-Zagier theorem (specialized to an elliptic curve). The above conjecture implies the Birch and Swinnerton-Dyer conjeture.

I think a Chebotarev argument should allow one to prove the following theorem:

Theorem 1.4 (Not proved yet)**.** *Conjecture 1.3 follows from the Manin constant conjecture, the BSD conjecture, and a refinement of Kolyvagin's conjecture, at least up to* 2 *and primes* p *where the mod* p *representation attached to* E *is not surjective.*

Part of the subtlety is that unlike in Kolyvagin's work, in the definition of V we consider subgroups W_ℓ that are defined using λ where the condition on each prime divisor of λ is divisibility by the integer M. In my previous conjecture and in all Kolyvagin's work, M is replaced by a fixed prime power divisor of M, which makes the condition on λ vastly less restrictive.

It will be very interesting to see if Conjecture 1.3 has any hope of being true at 2 or primes p where the mod p representation is reducible. It is natural enough of a conjecture, that it just might work.