2010-01-08 (1) | NTS | W. Stein

Using Heegner Points to Define a Canonical Subgroup of the Top Exterior Power of $E(K)$.

§1. Introduction

$E/\mathbb{Q}$ optimal elliptic curve, $r_{an}(E/\mathbb{Q}) \geq 1$, N = conductor

$K = \mathbb{Q}(\sqrt{D})$, $D \leq -5$, $(D,N)=1$, $p \mid N \Rightarrow p$ splits in K, $r_{an}(E^D/\mathbb{Q}) \leq 1$.

$n = r_{alg}(E/K)$: $\Lambda^n E(K)_{/tor} \xrightarrow{\hat{h}} \mathbb{R}$ height function.

$= $ top exterior power $\approx \mathbb{Z}$.

Goal: Use Heegner points to construct element $v_K \in \Lambda^n E(K)_{/tor}$

such that $\dfrac{L^{(*)}(E/K,1)}{\Omega_{E/K}} = \hat{h}(v_K)$, ← Leading coeff. of Taylor.

§2. Definitions.

$P_\lambda = D_\lambda[y_\lambda] \in \left(E(K_\lambda)/M E(K_\lambda)\right)^{Gal(K_\lambda/\mathbb{Q})}$

ℓ any prime inert in K. ; $f = r_{an}(E/\mathbb{Q}) - 1$.

$t = r_{an}(E/K)$.

$$\begin{tikzcd}
W_\ell \subseteq E(K) \arrow[r, "red_\ell \circ Tr_{K/\mathbb{Q}}"] \arrow[d] & \left(E(\mathbb{F}_{\ell^2})/M_\ell E(\mathbb{F}_{\ell^2})\right)^{\langle Frob \rangle} \\
X = \langle \overline{P}_\lambda : \lambda = p_1 \cdots p_f \rangle \subseteq &
\end{tikzcd}$$

$M_\ell = \gcd(a_\ell, \ell+1)$ ← not nec. a prime power.

each p_i s.t. $M_\ell \mid \gcd(a_{p_i}, p_i+1)$.

Maybe: $\#(E(\mathbb{F}_\ell)/a_\ell) \mid \#(E(\mathbb{F}_{p_i})/a_{p_i})$

Let $V_\ell = \Lambda^t W_\ell \subseteq \Lambda^t E(K)$.

Let $V = \bigcap_{\text{all inert } \ell} V_\ell \subseteq \Lambda^t E(K)$.

V is canonical — only depends on E, K.

Defn. is fairly natural as well.

§3: Questions and Observations:

Prop: $\mathrm{rank}(V) > 0 \Longrightarrow$ Kolyvagin's conjectures

proof: Use Chebotarev density argument & Kol. false $\Rightarrow \bar{\rho}_\lambda = 0 \Rightarrow \bigcap V_\ell \subseteq E^D(\mathbb{Q})$ and $t \geq 3$ so ...

Prop: $\mathrm{rank}(V) > 0 \Longrightarrow \Lambda^t E(K)$ has rank 1.

Proof: $\mathrm{rank}(V) > 0 \Rightarrow$ Kolyvagin's conjectures

$$\Downarrow$$

$$\mathrm{rank}(E(\mathbb{Q})) \leq f.$$

we know $\quad \mathrm{rank}(E^D(\mathbb{Q})) = r_{an}(E^D/\mathbb{Q}) \leq 1.$

$t = r_{an}(E/K) = f + r_{an}(E^D/\mathbb{Q})$, so $\mathrm{rank}(E(K)) \leq f + r_{an}(E^D/\mathbb{Q}) = t$

But $\mathrm{rank}(V) > 0 \Rightarrow \mathrm{rank}(E(K)) \geq t$ since $V \subseteq \Lambda^t(E(K))$.

So $\mathrm{rank}(E(K)) = t$, so $\Lambda^t E(K)$ has rank 1. ∎

Let v_K be gen for V

Question: $$\frac{L^{(*)}(E/K, 1)}{\Omega_{E/K}} = \hat{h}(v_K).$$ (possibly not quite right at 2..)

If $t = 1$ this is Gross-Zagier.

Hope: Manin constant conj + BSD conjecture + refined Kolyvagin + Chebotarev Density + (Weinstein / Howard)
$\Longrightarrow$ Conj (at least at odd primes p where $\bar{\rho}_{E,p}$ surjective)

Example: E 1001c1 $K = \mathbb{Q}(\sqrt{-40})$

$r_{alg}(E/\mathbb{Q}) = 2.$ $\#\text{Ш}_{an}(E^K) = 1.$

$\text{Ш}_{an}(E/\mathbb{Q}) = 1$ $c_7 = 2$, $c_{11} = 3$, $c_{13} = 2$

$E(\mathbb{Q})_{tor} = 1$

All $\bar{\rho}_{E,\ell}$ surjective.

ℓ	$M_\ell = \gcd(a_\ell, \ell+1)$
2	1
3	1
5	3
⋮	
19	4
47	8
89	9
⋮	
271	8
1231	16

$E(\mathbb{F}_{19})/4E(\mathbb{F}_{19}) \approx \mathbb{Z}/4\mathbb{Z}$

← not cyclic. (47)

← cyclic (271)

So maybe

$$V = W_{89} \cap W_{1231} \underset{12}{\overset{?}{\subseteq}} \textstyle\bigwedge^3 E(K) = \langle g \rangle$$

$$\| \quad \langle 12g \rangle \; ?$$