Take Home Midterm for Math 581F
Due Monday November 5, 2007, at the start of class.

There are five problems. Do not talk to other people about the problems, though you may email me if you have questions about problems (in case they are ill-stated). You may use Sage if you very explicitly describe exactly what calculations you did, and you may consult any book or web page (cite it). Make sure to do a good job on this exam, since it is *worth 30% of your grade*, almost the same as your entire homework. And it's not a terribly hard exam either. I fully expect everbody to get a perfect score.

1. (a) Consider the map $\varphi : \mathbb{Z}^3 \to \mathbb{Z}^4$ given by

   $$\varphi((1,0,0)) = (1,2,3,0), \quad \varphi((0,1,0)) = (4,5,6,0), \quad \varphi((0,0,1)) = (7,8,9,0).$$

   Write the cokernel of φ, i.e., $\mathbb{Z}^4/\operatorname{Im}(\varphi)$ as a direct sum of cyclic groups.

   (b) Which of the following rings are Noetherian?

   i. $\mathbb{Z}[\pi, e, \sqrt{2}]$ – adjoin some real numbers to $\mathbb{Z}$,
   ii. $\mathbb{F}_7(x_1, x_2, x_3)[y_1, y_2]$ – adjoin some variables to a finite field,
   iii. $\mathbb{Z}[x_1, x_2, \ldots, x_n, \ldots, n \geq 1]/(x_1 - x_2)$ – adjoin infinitely many variables to $\mathbb{Z}$ and quotient out by the ideal generated by $x_1 - x_2$,
   iv. $\overline{\mathbb{Z}}[x]$ – adjoin a variable to the ring of all algebraic integers in a fixed choice of algebraic closure of $\mathbb{Q}$.

2. (a) Prove that if K is a number field then there are infinitely many prime ideals of $\mathcal{O}_K$.

   (b) Suppose I and J are fractional ideals of a number field K.

   i. Prove that $I + J$ is a fractional ideal.
   ii. Prove that if $I = \prod \mathfrak{p}_i^{e_i}$ and $J = \prod \mathfrak{p}_i^{f_i}$, then $I + J = \prod_{i=1}^{r} \mathfrak{p}_i^{\min(e_i, f_i)}$.

3. Let $\mathcal{O}_K$ be the ring of integers of the number field $K = \mathbb{Q}(\sqrt{3^{997} - 1})$. [Note: You will probably not get anywhere trying to create this number field directly in the current version Sage.]

   (a) Find an order R such that $[\mathcal{O}_K : R]$ is not divisible by 7.

   (b) Give generators for the prime ideal factors of the ideal $7\mathcal{O}_K$.

4. Let $K = \mathbb{Q}(\sqrt{-23})$, and set $a = \sqrt{-23} \in K$.. Show how to find $n \in \mathbb{Z}$ and $\alpha \in \mathcal{O}_K$, such that

   $$\left(2a + 2, a - 11, -\frac{5}{2}a - \frac{17}{2}\right)\mathcal{O}_K = (n, \alpha)\mathcal{O}_K.$$

   Explain the steps you use to find n and α – don't just "ask Sage".

5. Make a list of every integral ideal I of the ring of integers of $K = \mathbb{Q}(\sqrt{-23})$ that has norm at most 10.