Homework 7 for Math 581F
Due Friday, November 30, 2007

Each problem has equal weight, and parts of problems are worth the same amount as each other.

1. Give a very detailed outline of your final project. **Your final project is due December 7, 2007.**

2. Let $K = \mathbb{Q}(\zeta_5)$ and let r be the number of real embeddings and s the number of pairs of complex conjugate embeddings.

   (a) Show that $r = 0$ and $s = 2$.

   (b) Find explicit generators for the group of units U_K.

   (c) Draw an illustration of the log map $\varphi : U_K \to \mathbb{R}^2$, including the hyperplane $x_1 + x_2 = 0$ and the lattice in the hyperplane spanned by the image of U_K.

3. Let $n = 6$. For a number field K, let e, f, g be the ramification, residue class degree, and number of primes over p for a rational prime p.

   (a) Give an example of a number field K of degree 6 and a prime p such that $e = 6$, or prove no such field exists.

   (b) Give an example of a number field K of degree 6 and a prime p such that $f = 6$, or prove no such field exists.

   (c) Give an example of a number field K of degree 6 and a prime p such that $g = 6$, or prove no such field exists.

   (d) Give an example of a number field K of degree 6 and a prime p such that $e = f = 2$, or prove no such field exists.

4. (a) Give an example of a finite nontrivial Galois extension K of $\mathbb{Q}$ and a prime ideal $\mathfrak{p}$ such that $D_{\mathfrak{p}} = \mathrm{Gal}(K/\mathbb{Q})$.

   (b) Give an example of a finite nontrivial Galois extension K of $\mathbb{Q}$ and a prime ideal $\mathfrak{p}$ such that $D_{\mathfrak{p}}$ has order 1.

   (c) Give an example of a finite Galois extension K of $\mathbb{Q}$ and a prime ideal $\mathfrak{p}$ such that $D_{\mathfrak{p}}$ is not a normal subgroup of $\mathrm{Gal}(K/\mathbb{Q})$.

   (d) Give an example of a finite Galois extension K of $\mathbb{Q}$ and a prime ideal $\mathfrak{p}$ such that $I_{\mathfrak{p}}$ is not a normal subgroup of $\mathrm{Gal}(K/\mathbb{Q})$.