Homework 5 for Math 581F
Due FRIDAY November 2, 2007

Each problem has equal weight, and parts of problems are worth the same amount as each other. This homework assignment has three problems that are quite easy, one that is very open ended, and two that might be difficult for you.

1. Let $\mathfrak{p}$ be one of the prime ideals that divides 2 in the ring of integers of $\mathbb{Q}(\sqrt{-13})$. Describe the abstract abelian group structure of the additive group of $\mathfrak{p}^{2006}/\mathfrak{p}^{2007}$.

2. (*) Give an example of an order $\mathcal{O}$ in the ring of integers of a number field and an ideal I such that I cannot be generated by 2 elements as an ideal. Does the Chinese Remainder Theorem hold in $\mathcal{O}$? [The (*) means that this problem is more difficult than usual.]

3. Let $a_1 = 1 + i$, $a_2 = 3 + 2i$, and $a_3 = 3 + 4i$ as elements of $\mathbb{Z}[i]$.

   (a) Prove that the ideals $I_1 = (a_1)$, $I_2 = (a_2)$, and $I_3 = (a_3)$ are coprime in pairs.

   (b) Compute $\#(\mathbb{Z}[i]/(I_1 I_2 I_3))$.

   (c) Find a single element in $\mathbb{Z}[i]$ that is congruent to n modulo I_n, for each $n \leq 3$. [Note: Sage doesn't have CRT over rings of integers yet, so you'll have to either do this problem partly by hand, which is definitely possible, or use another program such as PARI or Magma.]

4. Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_K$, and suppose that $\mathcal{O}_K/\mathfrak{p}$ is a finite field of characteristic $p \in \mathbb{Z}$. Prove that there is an element $\alpha \in \mathcal{O}_K$ such that $\mathfrak{p} = (p, \alpha)$. This justifies why we can represent prime ideals of $\mathcal{O}_K$ as pairs (p, α), as is done in SAGE.

5. (*) Use cyclotomic fields to prove that for every n there exists a number field F of degree n in which 2 splits comletely, i.e., factors as a product of n distinct primes in the ring of integers of F. Conclude that for every d there exists a number field whose ring of integers requires at least d generators. [Hint: See `http://wstein.org/129-05/challenges.html`, where a solution is given that is slightly sketchy, and that depends on learning some things about cyclotomic fields that I haven't covered in class. You may use theorems from other books or articles that we haven't proved in class or that aren't proved in the course textbook. Just give a precise citation for the theorem.]

6. Give a clear description that is less than a page long of what you intend to do your final project about.