Homework 3 for Math 581F, Due FRIDAY October 19, 2007

Each problem has equal weight, and parts of problems are worth the same amount as each other.

1. Prove that $\overline{\mathbb{Z}}$ is integrally closed in its field of fraction, and every nonzero prime ideal is maximal. Thus $\overline{\mathbb{Z}}$ is not a Dedekind domain only because it is not noetherian.

2. Let K be a field.
   (a) Prove that the polynomial ring $K[x]$ is a Dedekind domain.
   (b) Is $\mathbb{Z}[x]$ a Dedekind domain?

3. Prove that every finite integral domain is a field.

4. (a) Give an example of two ideals I, J in a commutative ring R whose product is *not* equal to the set $\{ab : a \in I, b \in J\}$.
   (b) Suppose R is a principal ideal domain. Is it always the case that
   $$IJ = \{ab : a \in I, b \in J\}$$
   for all ideals I, J in R?

5. Is the set $\mathbb{Z}[\frac{1}{2}]$ of rational numbers with denominator a power of 2 a fractional ideal?

6. Suppose you had the choice of the following two jobs[1]:

   Job 1 Starting with an annual salary of \$1000, and a \$200 increase every year.

   Job 2 Starting with a semiannual salary of \$500, and an increase of \$50 every 6 months.

   In all other respects, the two jobs are exactly alike. Which is the better offer (after the first year)? Write a Sage program that creates a table showing how much money you will receive at the end of each year for each job. (Of course you could easily do this by hand – the point is to get familiar with Sage.)

7. Let $\mathcal{O}_K$ be the ring of integers of a number field. Let F_K denote the abelian group of fractional ideals of $\mathcal{O}_K$.
   (a) Prove that F_K is torsion free.
   (b) Prove that F_K is not finitely generated.
   (c) Prove that F_K is countable.

[1] From *The Education of T.C. MITS* (1942).

(d) Conclude that if K and L are number fields, then there exists some (non-canonical) isomorphism of groups $F_K \approx F_L$.

8. From basic definitions, find the rings of integers of the fields $\mathbb{Q}(\sqrt{11})$ and $\mathbb{Q}(\sqrt{-6})$. Check your answers using Sage.

9. In this problem, you will give an example to illustrate the failure of unique factorization in the ring $\mathcal{O}_K$ of integers of $\mathbb{Q}(\sqrt{-6})$.

   (a) Give an element $\alpha \in \mathcal{O}_K$ that factors in two distinct ways into irreducible elements.

   (b) Observe explicitly that the (α) factors uniquely, i.e., the two distinct factorization in the previous part of this problem do not lead to two distinct factorization of the ideal (α) into prime ideals.

10. Factor the ideal (10) as a product of primes in the ring of integers of $\mathbb{Q}(\sqrt{11})$. You're allowed to use Sage, as long as you show the commands you use.