Homework 2 for Math 581F, Due FRIDAY October 12, 2007

Each problem has equal weight, and parts of problems are worth the same amount as each other.

1. Let $\varphi : R \to S$ be a homomorphism of (commutative) rings.
   (a) Prove that if $I \subset S$ is an ideal, then $\varphi^{-1}(I)$ is an ideal of R.
   (b) Prove moreover that if I is prime, then $\varphi^{-1}(I)$ is also prime.

2. Let $\mathcal{O}_K$ be the ring of integers of a number field. The Zariski topology on the set $X = \mathrm{Spec}(\mathcal{O}_K)$ of all prime ideals of $\mathcal{O}_K$ has closed sets the sets of the form
$$V(I) = \{\mathfrak{p} \in X : \mathfrak{p} \mid I\},$$
where I varies through *all* ideals of $\mathcal{O}_K$, and $\mathfrak{p} \mid I$ means that $I \subset \mathfrak{p}$.
   (a) Prove that the collection of closed sets of the form $V(I)$ is a topology on X.
   (b) Prove that the conclusion of (a) is still true if $\mathcal{O}_K$ is replaced by an order in $\mathcal{O}_K$, i.e., a subring that has finite index in $\mathcal{O}_K$ as a $\mathbb{Z}$-module.

3. Let $\alpha = \sqrt{2} + \frac{1+\sqrt{5}}{2}$.
   (a) Is α an algebraic integer?
   (b) Explicitly write down the minimal polynomial of α as an element of $\mathbb{Q}[x]$.

4. Which are the following rings are orders in the given number field.
   (a) The ring $R = \mathbb{Z}[i]$ in the number field $\mathbb{Q}(i)$.
   (b) The ring $R = \mathbb{Z}[i/2]$ in the number field $\mathbb{Q}(i)$.
   (c) The ring $R = \mathbb{Z}[17i]$ in the number field $\mathbb{Q}(i)$.
   (d) The ring $R = \mathbb{Z}[i]$ in the number field $\mathbb{Q}(\sqrt[4]{-1})$.

5. Give an example of each of the following, with proof:
   (a) A non-principal ideal in a ring.
   (b) A module that is not finitely generated.
   (c) The ring of integers of a number field of degree 3.
   (d) An order in the ring of integers of a number field of degree 5.
   (e) A non-diagonal matrix of left multiplication by an element of K, where K is a degree 3 number field.
   (f) An integral domain that is not integrally closed in its field of fractions.
   (g) A Dedekind domain with finite cardinality.
   (h) A fractional ideal of the ring of integers of a number field that is not an integral ideal.