Artin's Conjecture: Unconditional Approach and Elliptic Curve Analogue

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Artin's 'Primitive Root' Conjecture

Primitive Root

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An integer a is called the *primitive root* of a prime p if a generates the cyclic group $(\mathbb{Z}/p\mathbb{Z})^*$, i.e, the order of a modulo p is $p-1$.

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- Question 1
  "How many primitive roots are there for a fixed prime p?"

- Answer 1
  "If we fix p, there are $\phi(p-1)$ primitive root modulo p, where ϕ is the Euler totient function."

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How about reversing the question?

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What if we fix an integer *a* instead of fixing a prime *p* and ask a similar question?

- Question 2
  "If we fix an integer, 10 say, then for how many primes *p* will 10 be a primitive root?"

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- Question 2
  "If we fix an integer, 10 say, then for how many primes p will 10 be a primitive root?"

- Answer 2
  "10 is probably a primitive root for infinitely many primes p."

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Conjecture (Stronger Form)

If $a \neq 0, 1, -1$ and a is not a perfect square, then there exists a positive constant $A(a)$ depending on a such that for $x \to \infty$,

$$N_a(x) = \#\left\{p \leq x \ : \ \langle \overline{a} \rangle = (\mathbb{Z}/p\mathbb{Z})^*\right\} \sim A(a)\frac{x}{\log x}$$

where $\overline{a} = a \pmod{p}$.

Artin's Heuristic Idea

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$\langle \overline{a} \rangle = (\mathbb{Z}/p\mathbb{Z})^*$ if the following two events do not occur simultaneously for any prime q

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Value of a	$N_a(x)$	$A(a) \cdot \text{li}(x)$	% of Error
2	18701	17175	8.16
3	18761	17175	8.45
5	19699	17175	12.81
7	18687	17175	8.09
8	11225	17175	53.01
11	18772	17175	8.51

Here x is chosen to be the 50000-th prime, i.e, 611953.

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and for $\tilde{a} \equiv 1 \pmod 4$, *we have*

$$N_a(x) = C(h)\left(1 - \mu(|\tilde{a}|)\prod_{\substack{q|h \\ q|\tilde{a}}}\left(\frac{1}{q-2}\right)\prod_{\substack{q\nmid h \\ q|\tilde{a}}}\left(\frac{1}{q^2-q-1}\right)\right)\frac{x}{\log x} + O\left(\frac{x \log\log x}{\log^2 x}\right)$$

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Where

$$C(h) := \prod_{q \mid h} \left(1 - \frac{1}{q-1}\right) \prod_{q \nmid h} \left(1 - \frac{1}{q(q-1)}\right)$$

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Condition

The generalized Riemann hypothesis holds for the class of Dedekind zeta functions over Galois extensions of the type $\mathbb{Q}(\sqrt[k_1]{b}, \sqrt[k]{1})$, where $b \in \mathbb{Z}$, k is a square-free integer and $k_1|k$.

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- Figure out an unconditional proof !

Unconditional Proof of Artin's Conjecture

Gupta and Murty's Result

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Here,

$$N_a(x) \ = \ \#\{p \leq x \ : \ a \text{ is a primitive root of } p\}$$

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Notation: $(q, r, s)^u := q^{u_1} r^{u_2} s^{u_3}$ where $u := (u_1, u_2, u_3) \in \mathbb{Z}^3$ [non-negative].

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3. For each 2-dimensional subspace $V \subset \left(\frac{\mathbb{Z}}{2\mathbb{Z}}\right)^3$, any three elements of $S_V = \{u \in S_1 \;:\; u \not\equiv v \pmod 2 \;\forall\, v \in V\}$ are linearly independent

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The Set S_1

$$S_1 = \{(1,0,2), (3,2,0), (2,1,0), (0,3,2), (0,2,1), (2,0,3), (1,3,0), (3,1,2), (0,1,3), (2,3,1), (3,0,1), (1,2,3), (1,1,1)\}$$

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Lemma (Gupta and Murty)

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Important: Both the lemmas are true provided that $(p-1)$ has at most 3 odd prime divisors, all sufficiently large.

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Crucial Observation: The size of the set S in Gupta and Murty's theorem decreases if the previous lemma is strengthened by increasing the value of α.

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There are at most three square free integers greater than 1 for which Artin's conjecture does not hold.

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- And try to carry the information back to solve the original problem.

Elliptic Curve analogue of Artin's Conjecture

A brief introduction

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A general elliptic curve over $\mathbb{K}$ with char $\mathbb{K} \neq 2, 3$ can be written in the Weierstrass form $E : y^2 = x^3 + ax + b$ with $a, b \in \mathbb{K}$.

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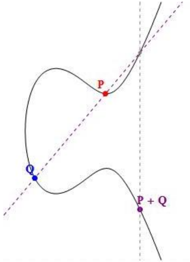

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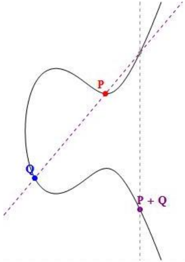

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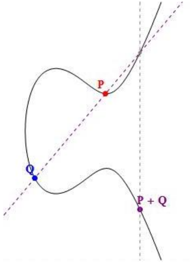

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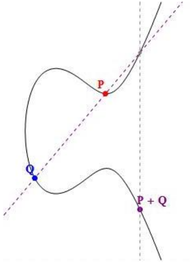

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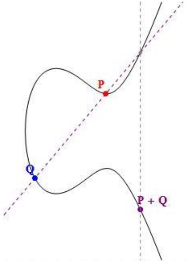

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- If a prime $p \nmid \Delta_E$, we can have a good (nonsingular) reduction of E modulo p to get $\overline{E}(\mathbb{F}_p)$ which also has an additive group structure.

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Put in mathematical notation, a rational point $a \in E(\mathbb{Q})$ is a *primitive point* of the curve modulo p if

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Theorem (Gupta and Murty, 1986)

Let $E(\mathbb{Q})$ be an elliptic curve defined over the rationals with complex multiplication by $\mathcal{O}_{\mathbb{K}}$ (entire ring of integers in an imaginary quadratic extension $\mathbb{K}$ over $\mathbb{Q}$) and let a be a rational point of infinite order.

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then under the assumption of generalized Riemann hypothesis, we obtain the following as $x \to \infty$:

$$N_a^*(x) \ = \ C_E(a)\frac{x}{\log x} \ + \ O\left(\frac{x \log \log x}{\log^2 x}\right)$$

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Hence we have a positive density in most of the cases.

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The assumption of GRH can be somewhat relaxed for higher rank case with E having complex multiplication.

Concluding remarks

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Open Questions: Elliptic Curve Analogue

- Is the analogous conjecture true unconditionally for all curves?
- Can we formulate the proof without the assumption of complex multiplication of the curve?
- Is the analogue in case of higher rank elliptic curves true without the assumption of GRH?

Questions ?

Thank you for your attention !