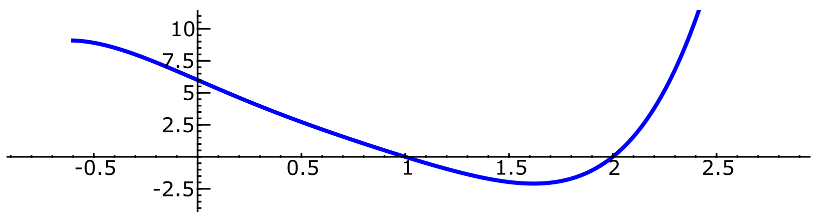
Exercises for Sections 5 and 6: Signature and Roots of Unity

Math 582e, Winter 2009, University of Washington

Due Wednesday February 18, 2009



1. Let $f = x^5 - 3x^4 + 3x^3 - 7x + 6$.
   (a) Find any *choice* of polynomials $A_0, A_1, \ldots, A_k \in \mathbb{R}[x]$ as in our algorithm from class (Section 4.1.2 of Cohen GTM 138) for computing the number of real roots of f.
   (b) Explicitly find the set Z of all real zeros of all these polynomials.
   (c) For each $x \in Z$, compute the sign sequence $A_0(x), A_1(x), \ldots, A_k(x)$.
   (d) How many real roots does f have?
2. (Exercise 25 of Chapter 4 of Cohen GTM 138) Let α be an algebraic integer of degree d all of whose conjugates have absolute value 1.
   (a) Show that for every positive integer k, the monic minimal polynomial of α^k in $\mathbb{Z}[X]$ has all of its coefficients bounded in absolute value by 2^d.
   (b) Deduce from this that there exists only a finite number of distinct powers of α, hence that α is a root of unity.
3. (Exercise 26 of Chapter 4 of Cohen GTM 138) Let $\rho \in \mathcal{O}_K$ be an algebraic integer given as a polynomial in θ, where $K = \mathbb{Q}(\theta)$ and T is the minimal monic polynomial of θ in $\mathbb{Z}[X]$. Give two algorithms to check exactly whether or not ρ is a root of unity, and compare their efficiency.
4. Show that the number field obtained by adjoing a root of x^4+1 to $\mathbb{Q}$ contains exactly 8 roots of unity. Do not simply run the Sage command `K.number_of_roots_of_unity()`.