

# Exercise Set 1:

## Matrix and Linear Algebra

Math 480b, Spring 2009, University of Washington

Due Wednesday, April 8, 2009

1. Given vectors  $x, y \in \mathbb{R}^n$ , the cosine of the angle  $\theta$  between them is  $\cos(\theta) = (x \cdot y) / (\|x\| \cdot \|y\|)$ , where  $\|(x_1, \dots, x_n)\| = \sqrt{\sum x_i^2}$ . Let  $x = (1, 2, 3, \dots, 100)$  and  $y = (1^2, 2^2, 3^2, \dots, 100^2)$ . What is the angle  $\theta$  between  $x$  and  $y$ ?
2. Plot the given 2 planes in 3D and a black line along their intersection:
  - (a)  $x_1 + x_2 + x_3 = 1$  and  $2x_1 + x_2 + 2x_3 = 1$ .
  - (b)  $x_1 - x_2 = 1$  and  $x_1 + x_2 + 2x_3 = 5$ .
3. Find the reduced row echelon form over the rational numbers  $\mathbb{Q}$  of the following matrices:
  - (a) The  $10 \times 10$  matrix whose entries are  $1, 2, 3, \dots, 97, 98, 99, 100$ , where  $1, \dots, 10$  are the first row,  $11, \dots, 20$  the second row, etc.
  - (b) The  $10 \times 10$  matrix whose  $i, j$  entry is  $i^2 + j^2$  for  $0 \leq i, j \leq 9$ . (In Sage, matrix indexing is 0-based.)
  - (c) The  $10 \times 10$  matrix whose  $i, j$  entry is  $i^2 \cdot j^2$  for  $0 \leq i, j \leq 9$ .
4. For which values of  $\alpha$  is the determinant of the following matrix equal to 0?

$$A = \begin{pmatrix} 1 & 2 & 0 & -\alpha \\ 1 & 0 & 0 & \alpha \\ \alpha & 0 & 1 & 2 \\ \alpha & 1 & -1 & 0 \end{pmatrix}$$

5.
  - (a) Create a random  $100 \times 100$  matrix  $A$  with double precision entries between 0 and 1. [Hint: use `random_matrix(RDF, 100, min=0, max=1)`.]
  - (b) Draw 2D plots of  $A$ ,  $A^{-1}$ ,  $A^2$ , and  $A^{100}$ .

- (c) Can you explain any of the patterns you see in your plots?
6. Make a Sage `interact` that takes a  $3 \times 3$  matrix  $A$  with entries in the rational field  $\mathbb{Q}$ , and computes and displays the following information about  $A$ :
- a plot of  $A$
  - the determinant of  $A$
  - the rank and nullity of  $A$
  - the reduced row echelon form of  $A$
  - the characteristic and minimal polynomials of  $A$
  - the eigenvalues and eigenvectors of  $A$
7. Let  $V = \text{span}((1/2, 0, 0), (0, 1, 1))$  and  $W = \text{span}((-1, 1, 0), (2/3, 1, 2))$ .
- (a) Plot  $V$  and  $W$  together in 3D.
- (b) Compute the sum  $V + W$  of  $V$  and  $W$ .
- (c) Compute the intersection  $V \cap W$ .
8. (a) Draw a plot in 2D of the set of  $(x, y) \in \mathbb{R}^2$  that satisfy

$$\begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 6$$

- (b) Draw a plot in 3D of the set of  $(x, y, z) \in \mathbb{R}^3$  that satisfy

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 2$$

Note that Sage (March 2009) does not have an `implicit_plot3d` command, so you will have to piece together plots using `plot_3d` or use `parametric_plot3d`.