FoCM 2011 Plenary

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Sage: A Community Owned

Foundation for Computational Mathematics

FoCM 2011 Plenary

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Testing, testing, ...

```
1 + 2 + 3
```

```
6
```

Prelude

A Dream Come True: Community-Owned Knowledge

- **Which do you use more?** Wikipedia or Encyclopædia Britannica?
- Which are you more likely to **contribute to**?
- The English Wikipedia alone has over 1 billion words, which is over 25 times as many as Encyclopædia Britannica.

Sage Project Mission Statement

"Create A Viable Free Open Source Alternative to Magma, Maple, Mathematica, and Matlab"

- **Mathematical features:** Of Magma, Maple, Mathematica, and Matlab, with comparable speed
- **Graphics:** 2d and 3d
- **Notebook:** Interactive graphical user interface
- **Documentation:** Books, papers, curriculum, etc.

Sage is not a drop-in replacement; it does not run programs written in the custom languages of the Ma's. It's nothing like Octave (versus Matlab). It's culture, architecture, and feel are all very different than the Ma's.

Why *not* Magma, Maple, Matlab, Mathematica?

1. **Commercial:** Expensive for my collaborators and students. Not community owned.
2. **Closed:** Implementation of algorithms often secret
3. **Frustrating:** Tight control of development
4. **Copy protection:** Hard to run on supercomputer or my new laptop
5. **Programming language:** All use a special math-only language
6. **Bugs:** Bug tracking is secret
7. **Compiler:** Lack of compilers for their math-only languages

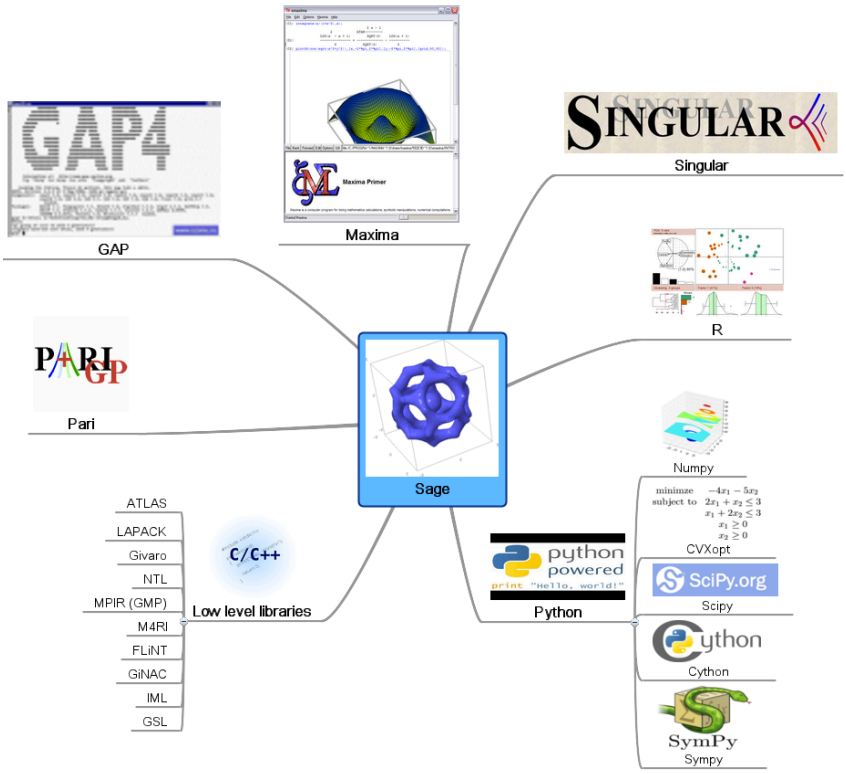
1. **Python:** a mainstream general purpose programming language (with a compiler: Cython)
2. **Distribution:** about 100 open source packages (**written by you** and your colleagues!)
3. **Interfaces:** smoothly tie together all these libraries and packages
4. **New library:** implements novel algorithms; well over a half million lines; written by a worldwide community of several hundred people.

```
R.<x,y> = PolynomialRing(QQ,2)
```

```
R
```

```
Multivariate Polynomial Ring in x, y over Rational Field
```

Distribution



Hundreds of Sage Developers

William Stein, Tim Abbott, Michael Abshoff, Antti Ajanki, Martin Albrecht, Nick Alexander, Bill Allombert, Ethan Van Andel, Ivan Andrus, Pablo Angulo, Benjamin Antieau, André Apitzsch, Maite Aranes, Oscar Gerardo Lazo Arjona, Eviatar Bach, Jennifer Balakrishnan, Jason Bandlow, Gregory Bard, Sébastien Barthélemy, Rob Beezer, Karim Belabas, Arnaud Bergeron, Luis Berlioz, Erin Beyerstedt, Francois Bissey, Jonathan Bober, Tom Boothby, Nicolas Borie, Johan Bosman, Robert Bradshaw, Michael Brickenstein, Nils Bruin, André-Patrick Bubel, Stanislav Bulygin, Dan Bump, Iftikhar Burhanuddin, Paul Butler, Oriol Castejón, Ondrej Certik, Wilson Cheung, Dan Christensen, Craig Citro, Anders Claesson, Francis Clarke, Timothy Clemans, Alex Clemesha, Nathann Cohen, Jenny Cooley, John Cremona, Karl-Dieter Crisman, Fidel Barrera Cruz, Doug Cutrell, Alyson Deines, Vincent Delecroix, Jeroen Demeyer, Tom Denton, Maarten Derickx, Didier Deshommes, Ryan Dingman, Dan Drake, Tom Draper, Alexander Dreyer, Tim Dumol, Nathan Dunfield, Gabriel Ebner, Ben Edwards, Dana Ernst, Burcin Erocal, Ron Evans, Richard J. Fateman, Lars Fischer, Jean-Pierre Flori, Evan Fosmark, Laurent Fousse, Gary Furnish, Alex Ghitza, Andrzej Giniewicz, Alain Giorgetti, Samuele Giraudo, Amy Glen, Daniel Gordon, Chris Gorecki, Jan Groenewald, Rob Gross, Jason Grout, Ryan Grout, Mathieu Guay-Paquet, Alexey U. Gudchenko, Harold Gutch, Jonathan Gutow, Jose Guzman, Anna Haensch, Carlo Hamalainen, Marshall Hampton, Jon Hanke, David Møllerller Hansen, Mike Hansen, Bill Hart, David Harvey, Leif Hille, Florent Hivert, Ryan Hinton, Neal Holtz, Golam Mortuza Hossain, Sean Howe, Alexander Hupfer, Wilfried Huss, Hamish Ivey-Law, Naqi Jaffery, Peter Jeremy, Peter Jipsen, Fredrik Johansson, Niles Johnson, Timo Jolivet, Benjamin Jones, David Joyner, Michael Kallweit, Josh Kantor, Kiran Kedlaya, Lloyd Kilford, Simon King, Keshav Kini, David Kirkby, Emily Kirkman, David Kohel, Ted Kosan, Ross Kyprianou, Sébastien Labbé, Yann Laigle-Chapuy, Kwankyu Lee, Julien Leroy, Richard Lindner, David Loeffler, Miguel Marco, Michael Mardaus, Robert Mařík, Jason Martin, Alexandre Blondin Massé, Peter McNamara, Gregory McWhirter, Jason Merrill, Matthias Meulien, Robert Miller, Kate Minola, Moritz Minzlaff, Joel Mohler, Thierry Monteil, Peter Mora, Bobby Moretti, Rich Morin, Guillaume Moroz, Gregg Musiker, Tobias Nagel, Brett Nakashima, Pablo De Nápoli, Johan Sebastian Rosenkilde Nielsen, Minh Van Nguyen, Andrey Novoseltsev, Christopher Olah, Johan Oudinet, Bill Page, Ronan Paixão, Willem Jan Palenstijn, John Palmieri, Dmitrii Pasechnik, Javier López Peña, Paulo César Pereira de Andrade, David Perkinson, Clement Pernet, John Perry, Pearu Peterson, David Poetzsch-Heffter, Viviane Pons, Bill Purvis, Julien Puydt, Yi Qiang, Jordi Quer, Gustavo Rama, Jens Rasch, Martin Raum, Dorian Raymer, Stefan Reiterer, R. Rishikesh, David Roe, Bjarke Hammersholt Roune, Gordon Royle, Serge A. Salamanka, Franco Saliola, Leonardo Sampaio, Kyle Schalm, Ed Scheinerman, Anne Schilling, Harald Schilly, Jack Schmidt, Michael Schneider, Christopher Schwan, Dag Sverre Seljebotn, Dan Shumow, Denis Simone, Steven Sivek, Nils-Peter Skoruppa, Jaap Spies, Jonathan Spreer, Armin Straub, Marco Streng, Kevin Stueve, Christian Stump, Blair Sutton, Chris Swierczewski, Luis Felipe Tabera Alonso, Glenn Tarbox, Philippe Theveny, Nicolas Thiery, Griffen Thoma, Emmanuel Thomé, John Thurber, Igor Tolkov, Gonzalo Tornaria, Kiminori Tsukazaki, Charlie Turner, Michel Vandenbergh, Joris Vankerschaver, Soledad Villar, John Voight, Felipe Voloch, Steve Vonn, Justin Walker, Mark Watkins, Georg S. Weber, Eric Webster, Ralf-Philipp Weinmann, Joe Wetherell, Carl Witty, Cristian Wuthrich, Soroosh Yazdani, Dal S. Yu, Gary Zablackis, Mike Zabrocki, Bin Zhang, Paul Zimmermann, Mao Ziyang

History

- **2005:**SAGE-0.1 released February 1, 2005; SAGE=Software for Arithmetic Geometry Experimentation
- **2006:** (2 Sage Days workshops); Sage is not just for number theory
- **2007:** (4 Sage Days) 100% test requirements; peer review of all new code (see trac); industry funding; NSF; Trophees du Libre
- **2008:** (7 Sage Days) Release managers besides me.

- **2009:** (8 Sage Days) Better foundations; 3d graphics; more developers (e.g., sage-combinat)
- **2010:** (13 Sage Days) More devs and users; nontrivial NSF grants
- **2011:** (>= 9 Sage Days) Faster http://sagenb.org; undergrad curriculum development taking off.

See this article for more details about the (pre-)history of Sage.

Question Break 1 (of 2)

????

```
reduce('2+3')
```

```
f = mathematica('Integrate[Sin[x],x]'); f
```
```
    -Cos[x]
```
```
f.Integrate(x)
```
```
    -Sin[x]
```
```
f + mathematica(sin(x)*cos(x^2))
```
```
    -Cos[x] + Cos[x^2]*Sin[x]
```
```
a = float(2^64); a
```
```
    1.8446744073709552e+19
```
```
math.sin(a)
```
```
    0.023598509904439558
```
```
mathematica('N[Sin[2^64], 100]')
```
```
    0.023598509904439558634365922876134775188797125047081359505208257512\
    787036719571480535326817566662323544
```
```
a = sin(2^64); a
```
```
    sin(18446744073709551616)
```
```
N(a, digits=100)
```

```
0.023598509904439558634365922876134775188797125047081359505208257512\
78703671957148053532681756666232354
```

```
a = [bernoulli(2*n) for n in [1..25]]; a
```

```
[1/6, -1/30, 1/42, -1/30, 5/66, -691/2730, 7/6, -3617/510,
43867/798, -174611/330, 854513/138, -236364091/2730, 8553103/6,
-23749461029/870, 8615841276005/14322, -7709321041217/510,
2577687858367/6, -263152715530534773731919190, 2929993913841559/6,
-2610827184964491220511353 0, 1520097643918070802691/1806,
-27833269579301024235023/690, 59645111159391216327796 1/282,
-560940336899781768624912754 7/46410, 49505720524107964821247752 5/66]
```

```
B = matrix(5,5,a); show(B)
```

$$\begin{pmatrix} \frac{1}{6} & -\frac{1}{30} & \frac{1}{42} & \\ -\frac{691}{2730} & \frac{7}{6} & -\frac{3617}{510} & \\ \frac{854513}{138} & -\frac{236364091}{2730} & \frac{8553103}{6} & -\frac{23}{2929993} \\ -\frac{7709321041217}{510} & \frac{2577687858367}{6} & -\frac{263152715530534773731919190}{} & \\ \frac{1520097643918070802691}{1806} & -\frac{27833269579301024235023}{690} & \frac{59645111159391216327796 1}{282} & -\frac{560940336899781768}{46410} \end{pmatrix}$$

```
maxima(B)
```

```
matrix([1/6,-1/30,1/42,-1/30,5/66],[-691/2730,7/6,-3617/510,43867/79\
8,-174611/330],[854513/138,-236364091/2730,8553103/6,-23749461029/87\
0,8615841276005/14322],[-7709321041217/510,2577687858367/6,-26315271\
553053477373/1919190,2929993913841559/6,-261082718496449122051/13530\
],[1520097643918070802691/1806,-27833269579301024235023/690,59645111\
1593912163277961/282,-560940336899781768624912754 7/46410,49505720524\
1079648212477525/66])
```

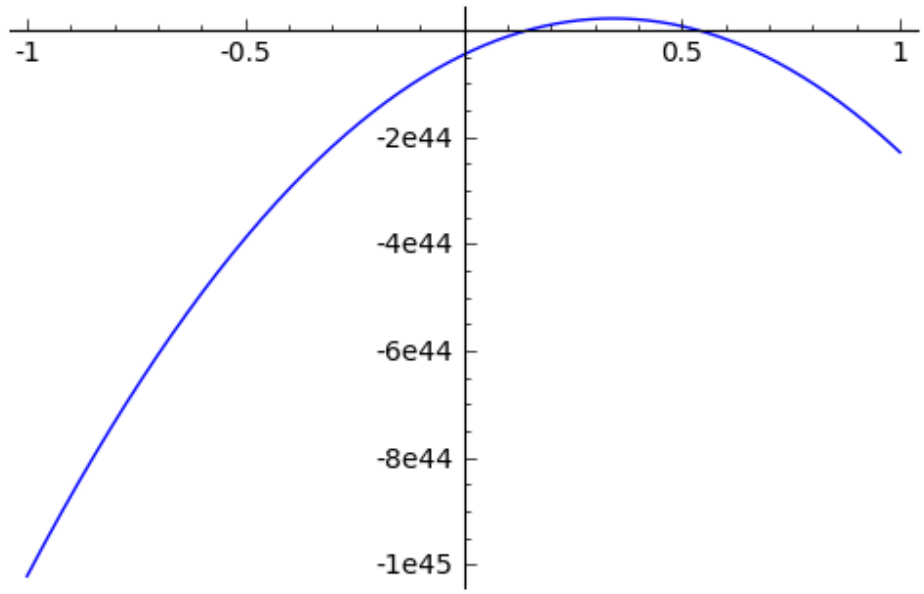
```
type(B)
```

```
<type
'sage.matrix.matrix_rational_dense.Matrix_rational_dense'>
```

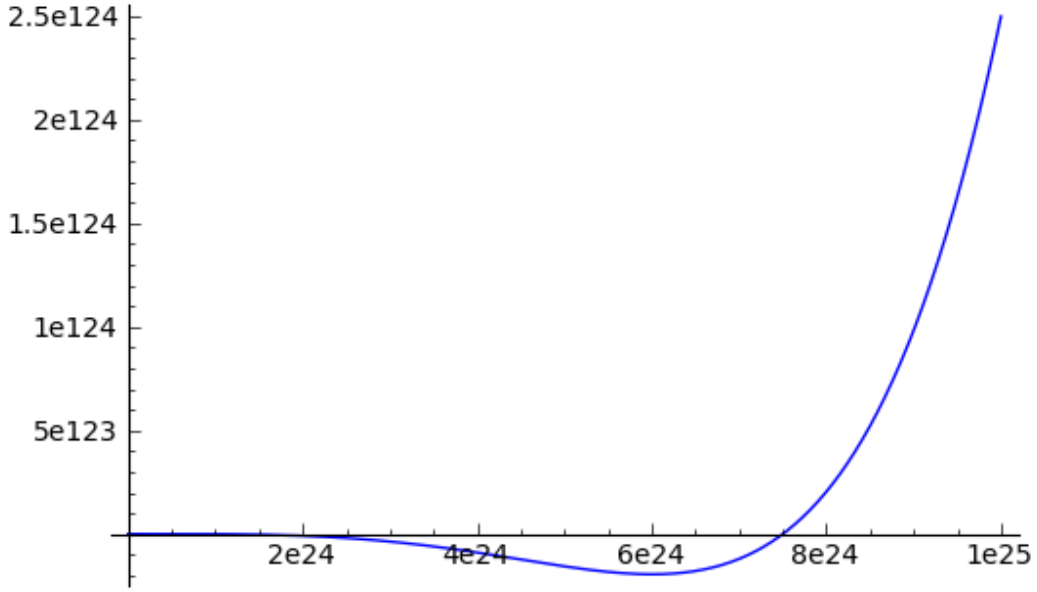
```
f = B.charpoly(); f
```

```
x^5 - 495057205273309581358818895/66*x^4 +
619832278402066658364047335791087369025854405274405715 08/46582559287\
006925*x^3 -
33295364506767959799223018166772957016217271828182269373583577643 36/\
57506448935165770954 05*x^2 +
64933335761993897773361542164718264243729556667127656381179337113 60/\
16375645934871014776439*x -
34317620809711478937719853025690725957250088159177607831953080320 00/\
73857505134826413583531
```

```
plot(f, 0,1e25)
```




```
plot(f, 0,1e25)
```




```
f.roots(RDF)
```

```
[(0.150059334256, 1), (0.534801951741, 1), (435126.664288, 1),
(1.77394190156e+14, 1), (7.50086674639e+24, 1)]
```

```
RDF
```

```
Real Double Field
```

```
a  = sqrt(2) + 5/3; a
```

```
sqrt(2) + 5/3
```

```
a
```

```
sqrt(2) + 5/3
```

```
parent(5/3)
```

```
Rational Field
```

```
parent(sqrt(2))
```

```
Symbolic Ring
```

```
N(a, 100)
```

```
    3.08088022903976171546835390876364745236338542043614739843346404657\
    39914512877370551705421?
```

```
v.lower()
```

```
    3.08088022903976171546835390876364745236338542043614739843346404657\
    39914512877370551705420
```

```
v.upper()
```

```
    3.08088022903976171546835390876364745236338542043614739843346404657\
    39914512877370551705421
```

```
RealField(300)
```

```
    Real Field with 300 bits of precision
```

```
f.roots(RealField(300))
```

```
    [(0.150059333481093391773686375578264240701638829767703759632823388\
    7\
    3668990898759929062125400\
    0, 1),
    (0.534801953521090603789758819043126592253340136980208644293456816\
    76\
    8698841917499312647873830, 1),
    (435126.66429725645426926118428433129795318655189156816174099426075\
    2\
    6523079651648826440803\
    04, 1),
    (1.773941901556805911055395566830335437216577242439932433095402584\
    38\
    5327063033430397171965\
    51e14, 1),
    (7.500866746387902496664023802160159411106133950973058780654411492\
    24\
    4167256150493349088041\
    57e24, 1)]
```

```
v = B.eigenvalues(); v
```

```
    [0.1500593334810934?, 0.5348019535210906?, 435126.6642972565?,
    1.773941901556806?e14, 7.500866746387902?e24]
```

```
alpha = v[0]; alpha
```

```
    0.1500593334810934?
```

```
alpha.minpoly()
```

```
    x^5 - 4950572052733095813588188\
    95/66*x^4 +
    6198322784020666583640473357910873690258544052744057150\
    8/46582559287\
    006925*x^3 -
    3329536450676795979922301816677295701621727182818226937358357764336/\
    5750644893516577095405*x^2 +
    6493333576199389777336154216471826424372955666712765638117933711360/\
    1637564593487101477643\
    9*x -
    3431762080971147893771985302569072595725008815917760783195308032000/\
    7385750513482641358353\
    1
```

```
```

Interactive Image Compression

(using numpy)

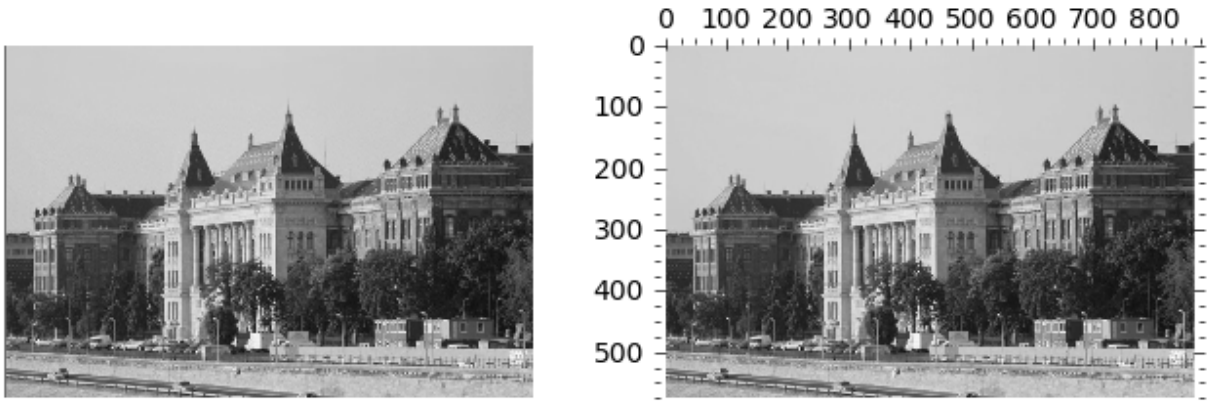
```
import pylab, numpy

X = pylab.imread(DATA + 'focm.png')
A_image = numpy.mean(X, 2)
u,s,v = numpy.linalg.svd(A_image)
S = numpy.zeros(A_image.shape)
S[:len(s),:len(s)] = numpy.diag(s)
n = A_image.shape[0]

@interact
def svd_image(i = ("Eigenvalues (quality)",
                   (20,(1..A_image.shape[0]//2)))):
    A_approx = numpy.dot(numpy.dot(u[:,:i], S[:i,:i]), v[:i,:])
    g = graphics_array([matrix_plot(A_approx),
                        matrix_plot(A_image)])
    show(g, axes=False, figsize=(8,4))
    html("Compressed to %.1f%% of size using %s eigenvalues."%(
                  100*(2.0*i*n+i)/(n*n), i))
```

Eigenvalues (quality) 20

```
Compressed to 78.2% of size using 225 eigenvalues.
```



Number Theory

```
factor(2009201020112012)      # uses PARI
```

```
    2^2 * 43 * 2269 * 5148259709
```

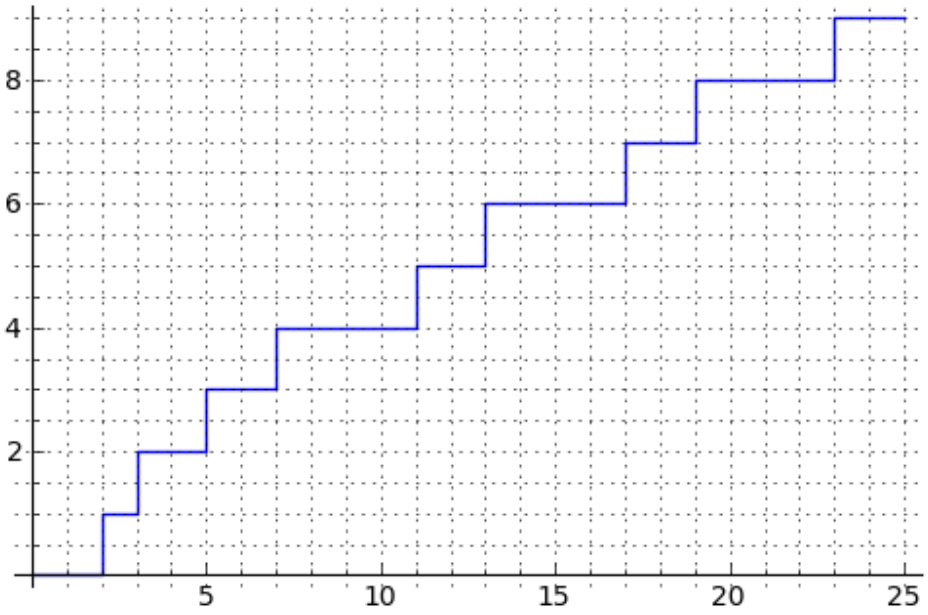
```
# Jon Bober - Rademacher's formula
time number_of_partitions(10^7)
```

```
920271755026045466855962781668256054307294052810239793953285763517 41\
298526232350197882291654710333933219876431112892669442996519201446 93\
371805742588542551019656697136927224393688612370494439001184662672 42\
229358838809496460215546742114497122936318794382420922229797018587 87\
035045131791561718499094276679781015502944193307504577212918898104 16\
144893435453842064389951868365922625931251702234301276800624906634 77\
743842242002004914231357899486287124678626100600066102278733540933 44\
771970346402912468019502617741296485750068965727678736574879683519 23\
635706131913486091452442762707644658047774085759494405085514475664 18\
811489630464191115045309280131652547731782793747141150484980319367 43\
061146399094602347281946619566715867818368113040887581799683872172 94\
457757539166632287129545104811207049238590872752415923922236658769 10\
286300131474621294645735699401736284697581755151900016413451408993 67\
093190859080267185611792170429465197857968117435299141079155705662 24\
961733691185950928557858441344173381696491426925891874353017426152 21\
458864914330678814703958326474085781955660425664604942849123726120 48\
505127243987254597660690323804252237148083121685300106473794690689 80\
174750466194629947200598144294809492676456385317172783153861091405 67\
421507374975384275021240249642090800336572787686941682199463463099 06\
686670220404292685540282106667661584147201155740243521818020629234 01\
192416721410067481998267445932984517612692068918153230357474682388 59\
771982766512554781263236924319788525197851803361705413947686093663 89\
114772406136683349412746708709224596935940110215929918473383014389 06\
563276480526765759086551359397824424431168085421773553659282321900 11\
995697704315727188881084528176142957772556227153624081222658008489 32\
276196267545354426556723347685735897551768373108977390874033016418 61\
826606886141268004692641729957436799211806554382447196471158568214 66\
186517301168907732792274321379544884573350740004120158204881150331 94\
548391813526724286716271750346470118187739995800431441608729698516 33\
242825634586248440662363995627333547254574113476278284725659303362 7\
186674151973687980402192233716371343312564266298874305195901960277 03\
231191899072357523708159719377437782971846471928181434133277347743 72\
834606272078694789174491151624899826575815641162980234597324952700 75\
351700956243598913730840133454971118265085585706676644886436331834 76\
426421938705419176750248730025527079093923132020478303851642054200 88\
883131037848381108842554875850837332409387347973027343769367934208 10\
867020185209718779670927269908884747550412550772822753523086679809 53\
624552474645295774554823817830271752015905175487656200680986962732 50\
359821161866711126479358328714703232879624446985159888486219011877 55\
088113589185457016501136471407349024158013999313263656227842088073 98\
682277421689531141855076998289838068746126554377321750652288128533 27\
497124638951381805103781784970476908891181848465927679498680646563 42\
862993393444237184906674722873849811294966439997475566061223014020 27\
411096367172771385437211572122448922846957076711812454645572143320 12\
897163090638216734538158656594031674831608270687758454094849618839 82\
333993500194238336824233008140594931590134175918949793856506343243 08\
841947277121816559279335938926911510683549043290690287102733571315 22\
276184648261543178606181345463363441597417941394202470612998672573 44\
715523461677386135094707607583386378705799210071685144173415484815 13\
953296373455058614174692678013759737246724696931125240457406888289 15\
405503038759354894288054926238362125959408069969864324535545382656 73\
785009637816816590962761268579690782176772889 80
Time: CPU 0.32 s, Wall: 0.50 s
```

```
@interact
```

```
def _(n=(25..10000)):
    plot(prime_pi, 0, n, gridlines='minor').show()
```

n 25



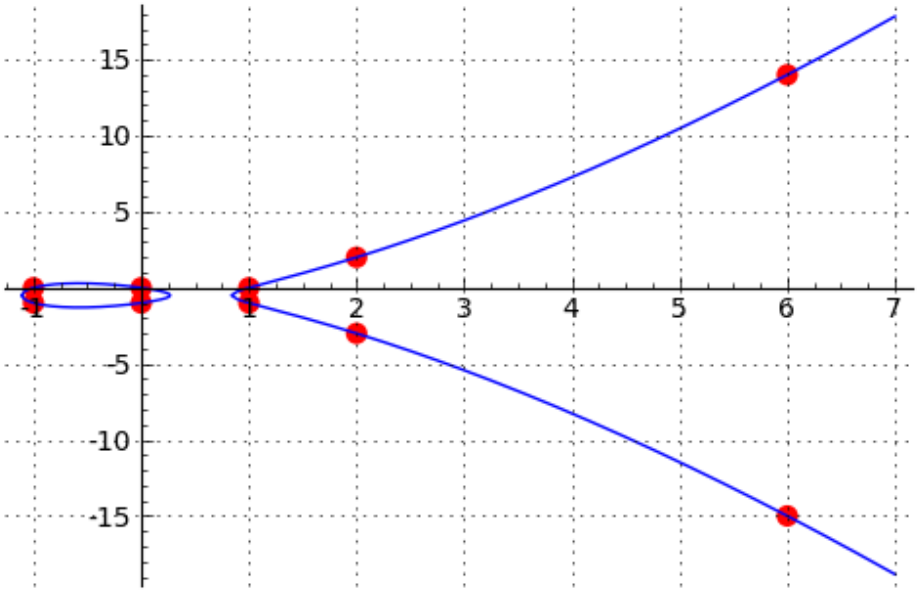

```
E = EllipticCurve('37a'); E
```

```
Elliptic Curve defined by y^2 + y = x^3 - x over Rational Field
```

```
v = E.integral_points(both_signs=True); v     # mwrank, new code
```

```
[(-1 : -1 : 1), (-1 : 0 : 1), (0 : -1 : 1), (0 : 0 : 1), (1 : -1 :
1), (1 : 0 : 1), (2 : -3 : 1), (2 : 2 : 1), (6 : -15 : 1), (6 : 14 :
1)]
```

```
plot(E, xmax=7, gridlines=True) + points([z[:2] for z in v],
          color='red', pointsize=50)
```

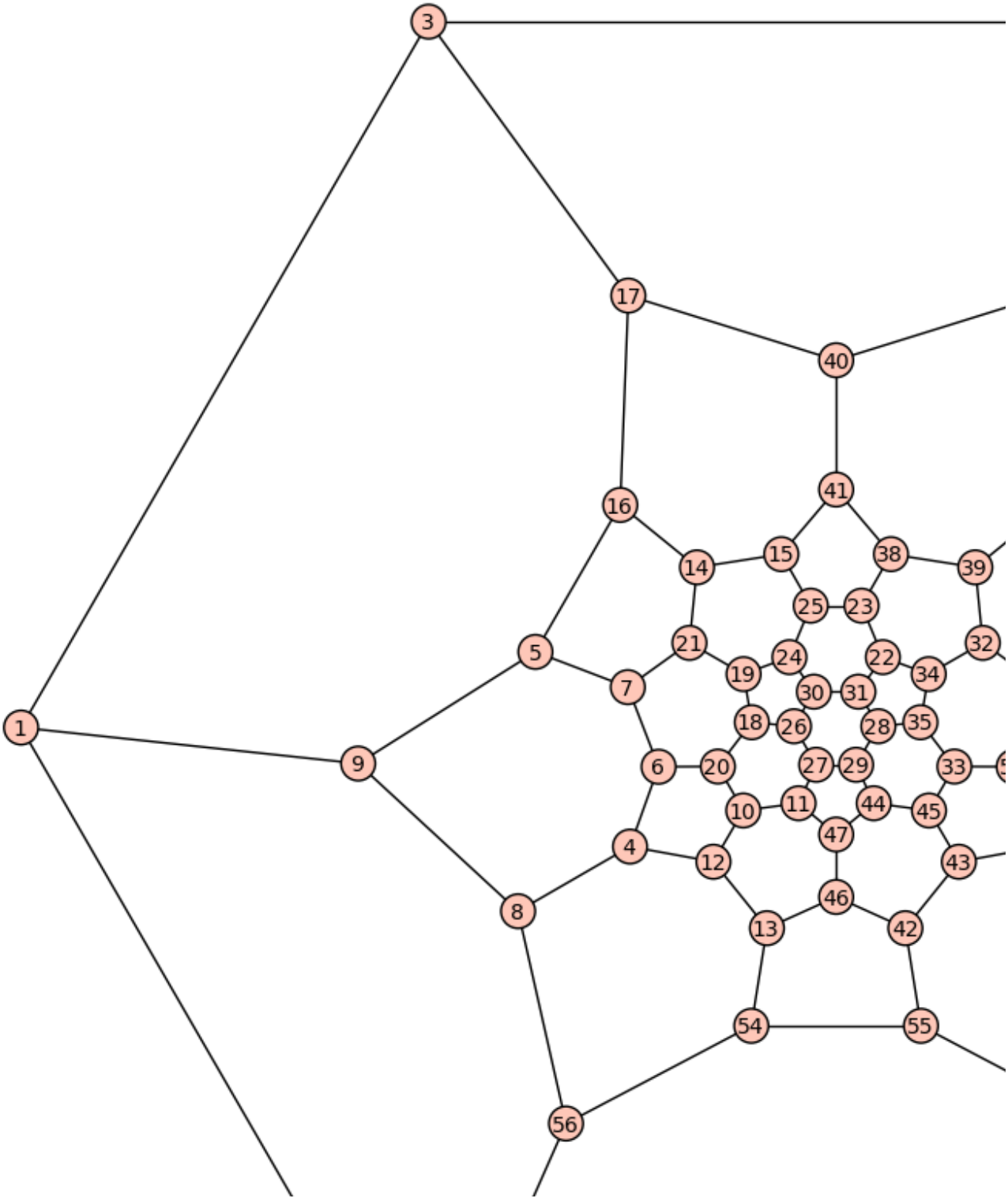


```
E.change_ring(GF(next_prime(10^50))).cardinality()   # pari SEA
```

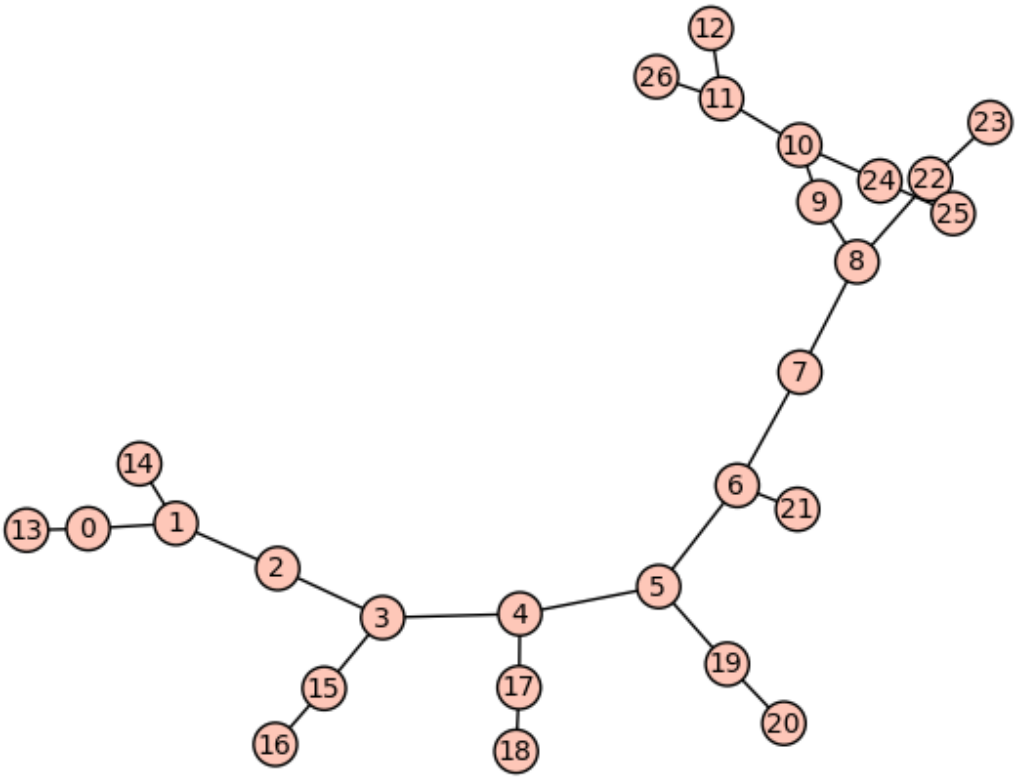
```
100000000000000000000000001917684156174529696959920
```

Graph Theory

```
graphs.BuckyBall().plot().show(figsize=15)
```




```
set_random_seed(1)
G = graphs.RandomLobster(8, .6, .3)
show(G, figsize=7)
```




```
G.automorphism_group()
```

```
    Permutation Group with generators [(12,26)]
```

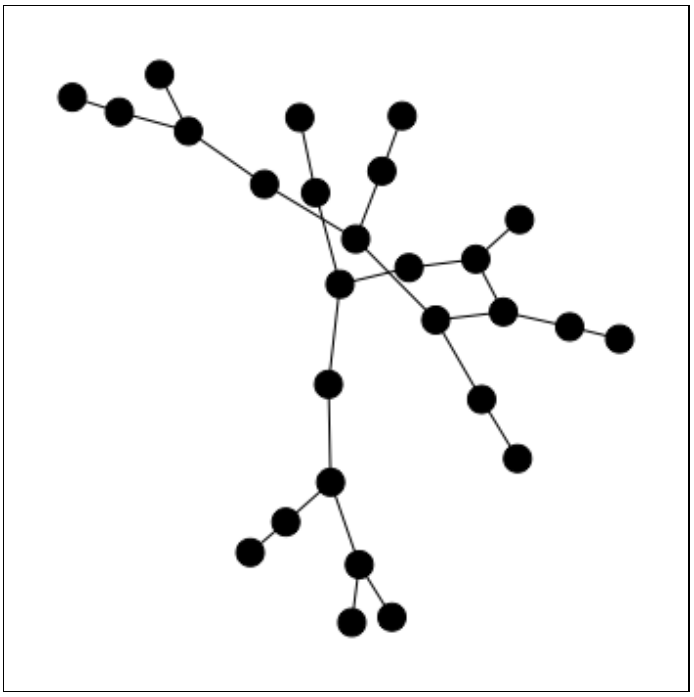
```
G.chromatic_number()
```

```
    2
```

```
G.shortest_path(13,20)
```

```
    [13, 0, 1, 2, 3, 4, 5, 19, 20]
```

```
graph_editor(G)
```



live:
variable name: G
strength:
length:
help

Save Close

```
graphs.
```

```
    <sage.graphs.graph_generators.GraphGenerators instance at 0x109f54368>
```

Cython

- Smooth transition between Python and compiled C code.
- Make code that involves lots of manipulation of C-level data structures optimally fast (orthogonal to algorithm choice).
- Heavily used in scientific computing using Python.

```
def python_sum2(n):
    s = int(0)
    for i in xrange(1, n+1):
        s += i*i
    return s
```

```
python_sum2(3)
```

```
    14
```

```
python_sum2(2*10^6)
```

```
2666668666667000000
```

```
timeit('python_sum2(2*10^6)')
```

```
5 loops, best of 3: 170 ms per loop
```

```
%cython
def cython_sum2(long n):
    cdef long i, s = 0
    for i in range(1, n+1):
        s += i*i
    return s
```

__Users_ws..._code_sage297_spyx.c __Users_ws...de_sage297_spyx.html

```
cython_sum2(3)
```

```
14
```

```
cython_sum2(2*10^6)
```

```
2666668666667000000
```

```
timeit('cython_sum2(2*10^6)')
```

```
125 loops, best of 3: 1.94 ms per loop
```

```
153/1.88
```

```
81.3829787234043
```

```
```

Of course, it is better to choose a different algorithm:

```
var('k, n')
show(factor(sum(k^2, k, 1, n)))
```

$$\frac{1}{6}(n+1)(2n+1)n$$

```
def sum2(n):
    return n*(2*n+1)*(n+1)/6
```

```
sum2(2*10^6)
```

```
2666668666667000000
```

```
timeit('sum2(2*10^6)')
```

```
625 loops, best of 3: 2.81 µs per loop
```

Even then, Cython provides a speedup:

```
%cython
def c_sum2(long n):
    return n*(2*n+1)*(n+1)/6
```

__Users_ws..._code_sage306_spyx.c __Users_ws...de_sage306_spyx.html

```
c_sum2(3)
```

```
14
```

```
timeit('c_sum2(2*10^6)')
```

```
625 loops, best of 3: 653 ns per loop
```

But at a cost!

```
c_sum2(2*10^6)        # WARNING: overflow -- it's just like C...
```

```
-407788678951258603
```

```
n=2*10^6; n*(2*n+1)*(n+1) > 2^63
```

```
True
```

```

```

Solving Equations

Solve a ***cubic equation***:

```
x = var('x'); show(solve(x^3 + x - 1==0, x)[0])
```

$$x = -\frac{1}{2}\left(i\sqrt{3}+1\right)\left(\frac{1}{18}\sqrt{3}\sqrt{31}+\frac{1}{2}\right)^{\left(\frac{1}{3}\right)} + \frac{-i\sqrt{3}+1}{6\left(\frac{1}{18}\sqrt{3}\sqrt{31}+\frac{1}{2}\right)^{\left(\frac{1}{3}\right)}}$$

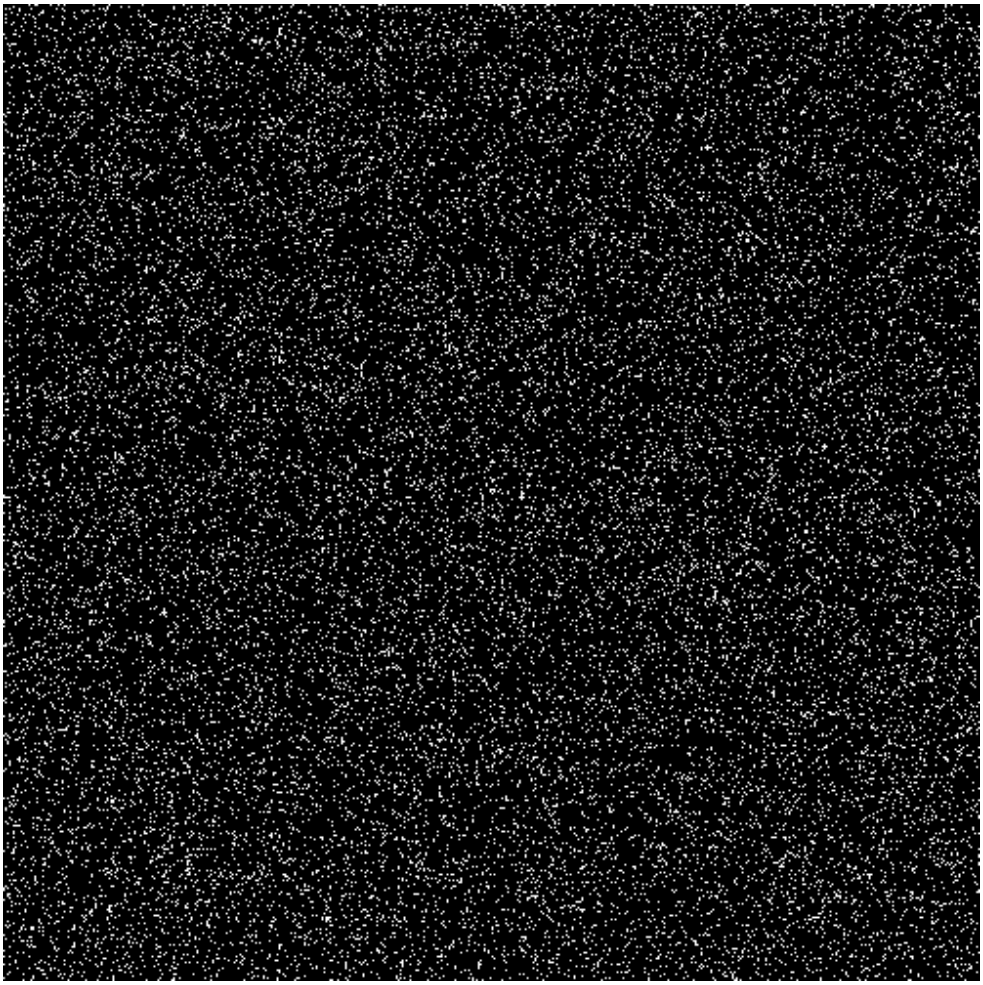
Solve a system of ***two linear equations*** with one unknown coefficient α:

```
var('alpha, y')
show(solve([3*x + 7*y == 2, alpha*x + 3*y == 8], x,y)[0])
```

$$\left[x = \frac{50}{7\,\alpha - 9}, y = \frac{2\,(\alpha - 12)}{7\,\alpha - 9}\right]$$

Solve a ***system of 500 linear equations*** exactly over the rational numbers:

```
n = 500
A = random_matrix(QQ,n,n,num_bound=10, den_bound=10)
v = random_matrix(QQ,n,1,num_bound=10, den_bound=10)
A.visualize_structure()
```



```
# IML is used -- http://www.cs.uwaterloo.ca/~astorjoh/iml.html
time w = A \ v
```

```
Time: CPU 2.08 s, Wall: 2.00 s
```

```
print w.str()
```

```
WARNING: Output truncated!
full_output.txt
```

```
[
-351702876068802906437552420686055850326076457904676962275100830127 5\
401420955261319182505472655403446871133375351213855350226815464192 62\
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752013079903081977572402942342855631847634917334375567725200949287 41\
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226942792117185857783063808851623050866691080434331157140193997647 05\
656405532372637424988128302930037221573841204291852109644817711181 71\
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```

$$
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6276519708370377748220107893500318506080007877295044619391333807678\
3
4404298127073292384584784066108783191909300052979739959990421818299\
2
2925855085253491955637538682896690137721577365833486970993840434691\
4
4344547635488272911439832922066075313432817036914368704456808129124\
5
3927322391051812751269413912650869583128167886051371218133372910572\
5
9043654960188499418451909765739706155032846814087502063072846200203\
5
6423014190813611899820786073110923736507592224098870834184189350561\
0
9384237418790130970924933437571099498573555767278861008062940814601\
2
0151715544497347937025971348203102252030841057764494331195246809761\
2
3866181969647041370142480229859354877697883210865623491026410955602\
2
0591159331838607574933592044009593318716045929933964320996096075895\
4401256445405736430656304065302152203845027370189816825023356066981\
5
4233159446864117915813842827734973170373721996301841763236329935471\
4
9438678327679504990114429522336377731017168239943148951267428379118\
7
2216663887150068039617770729890166569844868292547997093894548850663\
0
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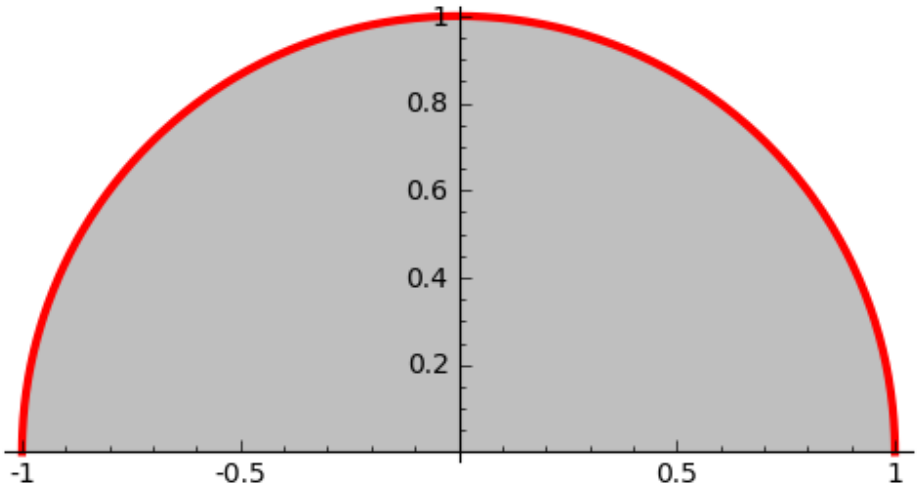
9
0982675778670635782913098070408133784285310000127724984673899778526\
8
7858238520181048432451889263811792822630864822089031199816712292749\
7
6649707235429823900190171387464844391690441164936391687051774383726\
6
2713949601230623140178323635887834977865301713317077639322919007773\
9
6289903101605022808141348530967880819658676298623903763230814863880\
8
1447209515623489102340]
```

full_output.txt

Symbolic Calculus

Symbolic Calculus makes use of **Maxima** and **Ginac** under the hood.

```
var('x')
f = sqrt(1 - x^2)
plot(f, thickness=3, color='red', aspect_ratio=1, fill=True)
```




```
var('t')
assume(t+1 > 0)
g = integrate(f, (x, -1, t)); show(g)
```

$$\frac{1}{4}\pi + \frac{1}{2}\sqrt{-t^2+1}t + \frac{1}{2}\arcsin(t)$$

```
show(g(t=1) - g(t=-1))
```

$$\frac{1}{2}\pi$$

```

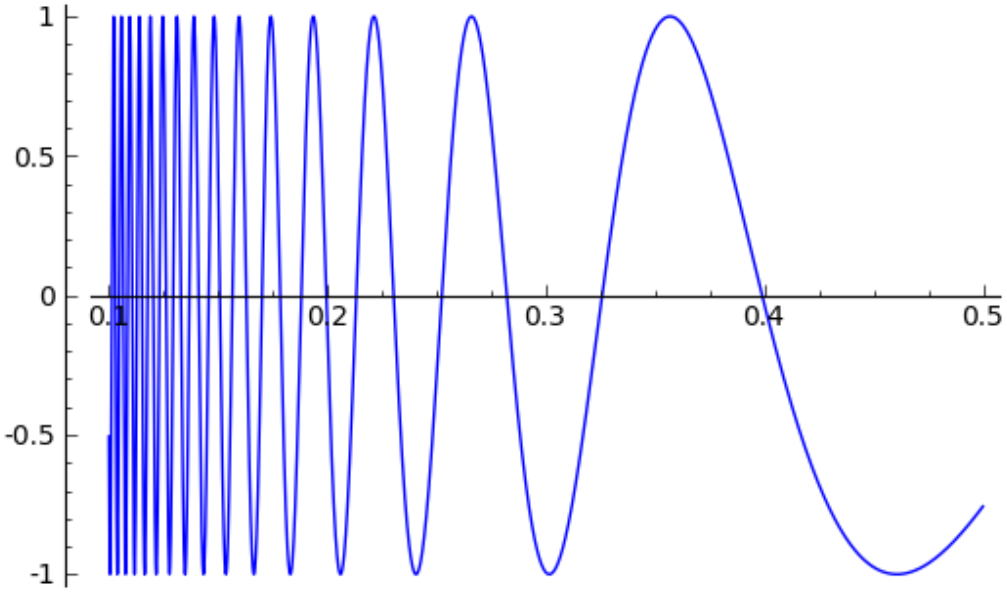
```

```

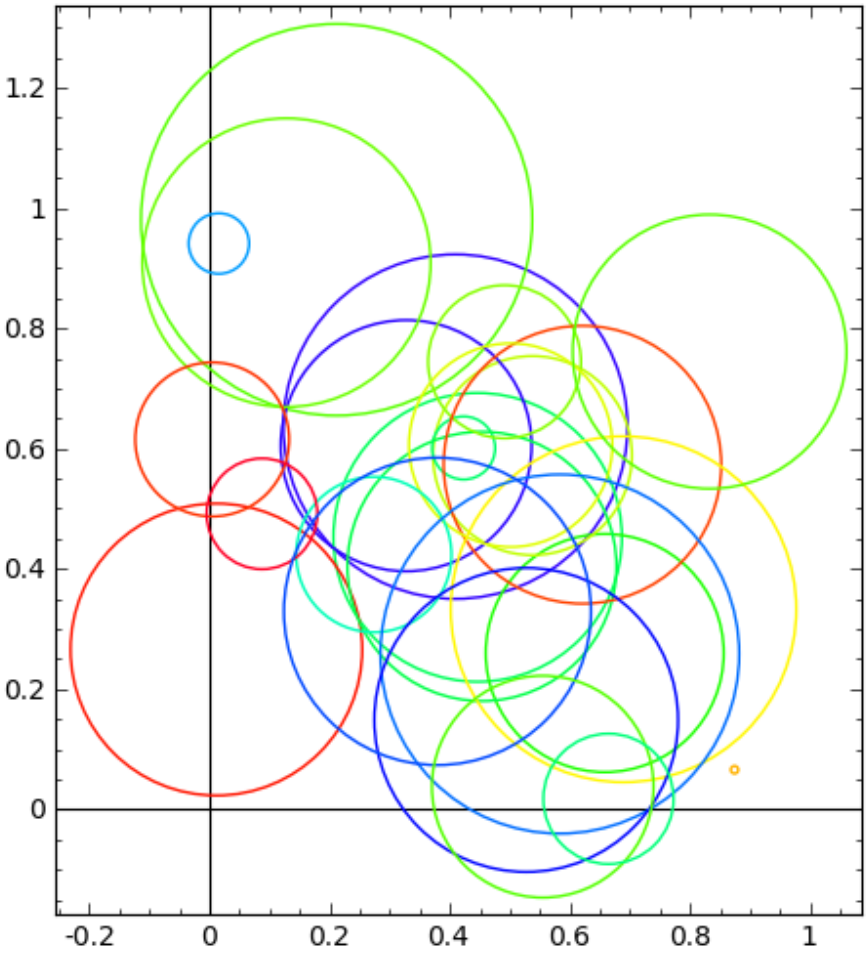
```

Plotting in 2D

```
plot(sin(1/x^2), (x,.1,.5))
```




```
G = sum(circle((random(), random()), random()/3,
        color=hue(random())) for i in range(25))
G.show(aspect_ratio=1, frame=True)
```



Plotting in 3D

```
f(x,y) = sin(x - y) * y * cos(x)
plot3d(f, (x,-3,3), (y,-3,3), opacity=.9, color='red')
```

Get Image

```
G = sum(sphere((random(), random(), random()), random()/4,
        color=hue(random()), opacity=.6)
        for i in range(20))
G.show(aspect_ratio=1, frame=True)
```

Get Image

Plotting a 3D Model

See http://www.davidson.edu/math/chartier/Starwars/

```
# Yoda! (53,756 vertices)

from scipy import io
yoda = io.loadmat(DATA + 'yodapose.mat')

from sage.plot.plot3d.index_face_set import IndexFaceSet
V = yoda['V']; F3=yoda['F3']-1; F4=yoda['F4']-1
Y = IndexFaceSet(F3,V,color=Color('#444444')) +
IndexFaceSet(F4,V,color=Color('#007700'))
Y = Y.rotateX(-1)
Y.show(aspect_ratio=1, frame=False, zoom=1.2)
```

Get Image

Question Break 2 (of 2)

????