

# MODULAR PARAMETRIZATIONS OF NEUMANN–SETZER ELLIPTIC CURVES

WILLIAM STEIN AND MARK WATKINS

ABSTRACT. Suppose  $p$  is a prime of the form  $u^2 + 64$  for some integer  $u$ , which we take to be  $3 \pmod{4}$ . Then there are two Neumann–Setzer elliptic curves  $E_0$  and  $E_1$  of prime conductor  $p$ , and both have Mordell–Weil group  $\mathbf{Z}/2\mathbf{Z}$ . There is a surjective map  $X_0(p) \xrightarrow{\pi} E_0$  that does not factor through any other elliptic curve (i.e.,  $\pi$  is optimal), where  $X_0(p)$  is the modular curve of level  $p$ . Our main result is that the degree of  $\pi$  is odd if and only if  $u \equiv 3 \pmod{8}$ . We also prove the prime-conductor case of a conjecture of Glenn Stevens, namely that if  $E$  is an elliptic curve of prime conductor  $p$  then the optimal quotient of  $X_1(p)$  in the isogeny class of  $E$  is the curve with minimal Faltings height. Finally we discuss some conjectures and data about modular degrees and orders of Shafarevich–Tate groups of Neumann–Setzer curves.

## 1. INTRODUCTION

Let  $p$  be a prime of the form  $u^2 + 64$  for some integer  $u$ , which we take to be  $3 \pmod{4}$ . Neumann and Setzer [Neu71, Set75] considered the following two elliptic curves of conductor  $p$  (note that Setzer chose  $u \equiv 1 \pmod{4}$  instead):

$$(1.1) \quad E_0 : y^2 + xy = x^3 - \frac{u+1}{4}x^2 + 4x - u,$$

$$(1.2) \quad E_1 : y^2 + xy = x^3 - \frac{u+1}{4}x^2 - x.$$

For  $E_1$  we have  $c_4 = p - 16$  and  $c_6 = u(p + 8)$  with  $\Delta = p = u^2 + 64$ , while for  $E_0$  we have  $c_4 = p - 256$  and  $c_6 = u(p + 512)$  with  $\Delta = -p^2$ . Thus each  $E_i$  is isomorphic to a curve of the form  $y^2 = x^3 - 27c_4x - 54c_6$  for the indicated values of  $c_4$  and  $c_6$ . The curves  $E_0$  and  $E_1$  are 2-isogenous and one can show using Lutz–Nagell and descent via 2-isogeny that

$$E_0(\mathbf{Q}) = E_1(\mathbf{Q}) = \mathbf{Z}/2\mathbf{Z}.$$

Moreover, if  $E$  is any elliptic curve over  $\mathbf{Q}$  of prime conductor with a rational point of order 2 then  $E$  is a Neumann–Setzer curve or has conductor 17 (see [Set75]).

Let  $X_0(p)$  be the modular curve of level  $p$ . By [Wil95] there is a surjective map  $\pi : X_0(p) \rightarrow E_0$ , and by [MO89, §5, Lem. 3] we may choose  $\pi$  to be optimal, in the sense that  $\pi$  does not factor through any other elliptic curve. The *modular degree* of  $E_0$  is  $\deg(\pi)$ .

We prove in Section 2 that the modular degree of  $E_0$  is odd if and only if  $u \equiv 3 \pmod{8}$ . Our proof relies mostly on results from [Maz77]. In Section 3 we show that  $E_1$  is the curve of minimal Faltings height in the isogeny class  $\{E_0, E_1\}$  of  $E_1$  and prove that  $E_1$  is an optimal quotient of  $X_1(p)$ , which is enough to prove the prime-conductor case of a conjecture of [Ste89] (this case is not covered by the results of [Vat03]). Finally, in Section 4 we give evidence for our conjecture that there are infinitely many elliptic curves with odd modular degree, and give a conjectural refinement of

Theorem 2.1. We also present some data about  $p$ -divisibility of conjectural orders of Shafarevich–Tate groups of Neumann–Setzer curves.

1.1. **Notation.** Let  $p$  be a prime and  $n$  be the numerator of  $(p - 1)/12$ .

We use standard notation for modular forms, modular curves, and Hecke algebras, as in [DI95] and [Maz77]. In particular, let  $X_0(p)$  be the compactified coarse moduli space of elliptic curves with a cyclic subgroup of order  $p$ . Then  $X_0(p)$  is an algebraic curve defined over  $\mathbf{Q}$ . Let  $J = J_0(p)$  be the Jacobian of  $X_0(p)$ , and let  $\mathbf{T} = \mathbf{Z}[T_2, T_3, \dots] \subset \text{End}(J)$  be the Hecke algebra. Also, let  $X_1(p)$  be the modular curve that classifies isomorphism classes of pairs  $(E, P)$ , where  $P \in E$  is a point of order  $p$ .

To each newform  $f \in S_2(\Gamma_0(p))$ , there is an associated abelian subvariety  $A = A_f \subset J_0(p)$ . We call the kernel  $\Psi_A$  of the natural map  $A \hookrightarrow J \rightarrow A^\vee$  the *modular kernel*. For example, when  $A$  is an elliptic curve, this map is induced by pullback followed by push forward on divisors and  $\Psi_A$  is multiplication by  $\deg(X_0(p) \rightarrow A)$ . The *modular degree* of  $A$  is the square root of the degree of  $\Psi_A$ . This definition makes sense even when  $\dim(A) > 1$ , since the degree of a polarization is the square of its Euler characteristic, hence a perfect square (see [Mum70, §16, pg. 150]). If  $I \subset \mathbf{T}$  is an ideal, let

$$A[I] = \{x \in A(\overline{\mathbf{Q}}) : Ix = 0\} \quad \text{and} \quad A[I^\infty] = \bigcup_{n>0} A[I^n].$$

1.2. **Acknowledgements.** The authors would like to thank the American Institute of Mathematics for hospitality while they worked on this paper, the National Science Foundation for financial support, and Matt Baker, Frank Calegari, and Barry Mazur for helpful conversations.

## 2. DETERMINATION OF THE PARITY OF THE MODULAR DEGREE

Let  $p, E_0, J$  and  $n$  be as in Section 1, and fix notation as in Section 1.1. In this section we prove the following theorem.

**Theorem 2.1.** *The modular degree of  $E_0$  is odd if and only if  $u \equiv 3 \pmod{8}$ .*

In order to prove the theorem we deduce seven lemmas using techniques and results from [Maz77].

Let  $m$  be the modular degree of  $E_0$ , and let

$$B = \ker(J \xrightarrow{\pi} E_0).$$

**Lemma 2.2.** *We have  $m^2 = \#(B \cap E_0)$ .*

*Proof.* As mentioned in Section 1.1, the composition  $E_0 \rightarrow J \rightarrow E_0$  is multiplication by the degree of  $X_0(p) \rightarrow E_0$ , i.e., multiplication by the modular degree of  $E_0$ . The lemma follows since multiplication by  $m$  on  $E_0$  has degree  $m^2$ .  $\square$

The *Eisenstein ideal*  $\mathcal{I}$  of  $\mathbf{T}$  is the ideal generated by  $T_\ell - (\ell + 1)$  for  $\ell \neq p$  and  $T_p - 1$ . By hypothesis, there is a Neumann–Setzer curve of conductor  $p$ , which implies that the numerator  $n$  of  $(p - 1)/12$  is even (we do the elementary verification that this numerator is even in the proof of Theorem 2.1 below). As discussed in [Maz77, Prop. II.9], the 2-*Eisenstein prime*  $\mathfrak{m} = (2) + \mathcal{I}$  of  $\mathbf{T}$  is a maximal ideal of  $\mathbf{T}$ , with  $\mathbf{T}/\mathfrak{m} \cong \mathbf{Z}/2\mathbf{Z}$ .

**Lemma 2.3.** *We have  $E_0[\mathfrak{m}] = E_0[2]$ .*

*Proof.* By [Maz77, Prop. II.11.1, Thm. III.1.2], the Eisenstein ideal  $\mathcal{I}$  annihilates  $J(\mathbf{Q})_{\text{tor}}$ , so  $\mathfrak{m}$  annihilates  $J(\mathbf{Q})_{\text{tor}}[2]$ . Since  $J(\mathbf{Q})_{\text{tor}}$  is cyclic of order  $n$  (by [Maz77, Thm. III.1.2]),  $J(\mathbf{Q})_{\text{tor}}[2]$  has order 2, so  $J(\mathbf{Q})_{\text{tor}}[2] = E_0(\mathbf{Q})_{\text{tor}}[2]$ , hence  $E_0(\mathbf{Q})[\mathfrak{m}] \neq 0$ . The Hecke algebra  $\mathbf{T}$  acts on  $E_0$  through  $\text{End}(E_0) \cong \mathbf{Z}$ , so each element of  $\mathbf{T}$  acts on  $E_0$  as an integer; in particular, the elements of  $\mathfrak{m}$  all act as multiples of 2 (since  $E_0[\mathfrak{m}] \neq 0$  and  $2 \in \mathfrak{m}$ ), so  $E_0[\mathfrak{m}] = E_0[2]$  since  $2 \in \mathfrak{m}$ .  $\square$

**Lemma 2.4.** *Suppose  $A \subset J_0(p)$  is a  $\mathbf{T}$ -stable abelian subvariety and  $\wp \subset \mathbf{T}$  is a maximal ideal such that  $A[\wp^\infty] \neq 0$ . Then  $A[\wp] \neq 0$ . Also  $A[\wp^\infty]$  is infinite.*

*Proof.* Arguing as in [Maz77, §II.14, pg. 112], we see that for any  $r$ ,  $A[\wp^r]/A[\wp^{r+1}]$  is isomorphic to a direct sum of copies of  $A[\wp]$ . If  $A[\wp] = 0$ , then since  $A[\wp^\infty] \neq 0$ , there must exist an  $r$  such that  $A[\wp^r]/A[\wp^{r+1}] \neq 0$ . But  $A[\wp^r]/A[\wp^{r+1}]$  is contained in a direct sum of copies of  $A[\wp] = 0$ , which is a contradiction.

To see that  $A[\wp^\infty]$  is infinite, note that if  $\ell$  is the residue characteristic of  $\wp$  and  $\text{Tate}_\ell(A)$  is the Tate module of  $A$  at  $\ell$ , then

$$\text{Tate}_\wp(A) = \varprojlim_r A[\wp^r] = \text{Tate}_\ell(A) \otimes_{\mathbf{T}} \mathbf{T}_\wp$$

is infinite. (For more details, see the proof of [RS01, Prop. 3.2].)  $\square$

The analogues of Lemmas 2.5–2.7 below are true, with the same proofs, for  $\mathfrak{m}$  any Eisenstein prime. We state and prove them for the 2-Eisenstein prime, since that is the main case of interest to us. Let  $\tilde{J}^{(2)}$  be the 2-Eisenstein quotient of  $J$ , where  $\tilde{J}^{(2)}$  is as defined in [Maz77, §II.10]. More precisely, we have the following:

**Lemma 2.5.** *The simple factors of  $\tilde{J}^{(2)}$  correspond to the  $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ -conjugacy classes of newforms  $f$  such that  $A_f[\mathfrak{m}] \neq 0$  (or equivalently,  $A_f^\vee[\mathfrak{m}] \neq 0$ ).*

*Proof.* On page 97 of [Maz77] we find that the  $\mathbf{C}$ -simple factors of  $\tilde{J}^{(2)}$  are in bijection with the irreducible components  $\text{Spec}(I_f)$  of  $\text{Spec}(\mathbf{T})$  which meet the support of the ideal  $\mathfrak{m}$ , so the  $I_f$  are the newform ideals contained in  $\mathfrak{m}$ . We have for any  $I_f$ ,

$$I_f \subset \mathfrak{m} \iff \mathbf{T}_m/(I_f)_m \neq 0 \iff \text{Tate}_m(A_f) \neq 0 \iff A_f[\mathfrak{m}] \neq 0.$$

Note that the same argument applies to  $A_f^\vee$ .  $\square$

**Lemma 2.6.** *Suppose  $A$  and  $B$  are abelian varieties equipped with an action of the Hecke ring  $\mathbf{T}$  and that  $\varphi : A \rightarrow B$  is a  $\mathbf{T}$ -module isogeny. If  $\wp \subset \mathbf{T}$  is a maximal ideal and  $B[\wp] \neq 0$ , then also  $A[\wp] \neq 0$ .*

*Proof.* Let  $\psi : B \rightarrow A$  be the isogeny complementary to  $\varphi$ , so  $\psi$  is the unique isogeny such that  $\psi \circ \varphi$  is multiplication by  $\deg(\varphi)$ . Then  $\psi$  is also a  $\mathbf{T}$ -module homomorphism (one can see this in various ways; one way is to use the rational representation on homology to view the endomorphisms as matrices acting on lattices, and to note that if matrices  $M$  and  $N$  commute, then  $M^{-1}$  and  $N$  also commute). By Lemma 2.4, the union  $B[\wp^\infty]$  is infinite, so  $\psi(B[\wp^\infty]) \neq 0$ . Since  $\psi(B[\wp^\infty]) \subset A[\wp^\infty]$ , Lemma 2.4 implies that  $A[\wp] \neq 0$ , as claimed.  $\square$

**Lemma 2.7.** *Suppose  $B \subset J_0(p)$  is a sum of abelian subvarieties  $A_f$  attached to newforms. If  $B[\mathfrak{m}] \neq 0$ , then there is some  $A_f \subset B$  such that  $A_f[\mathfrak{m}] \neq 0$ .*

*Proof.* There is something to be proved because if  $x \in B[\mathfrak{m}]$  it could be the case that  $x = y + z$  with  $y \in A_f$  and  $z \in A_g$ , but  $x \notin A_h$  for any  $h$ . Let  $C = \bigoplus A_f$ , where the  $A_f \subset J_0(p)$  are simple abelian subvarieties of  $B$  corresponding to conjugacy classes of newforms. Then there is an isogeny  $\varphi : C \rightarrow B$  given by

$$\varphi(x_1, \dots, x_n) = x_1 + \dots + x_n,$$

where the sum is in  $B \subset J_0(p)$ . By Lemma 2.6,  $C[\mathfrak{m}] \neq 0$ . Since  $C[\mathfrak{m}] \cong \bigoplus A_f[\mathfrak{m}]$ , it follows that  $A_f[\mathfrak{m}] \neq 0$  for some  $A_f \subset B$ .  $\square$

**Lemma 2.8.** *If  $4 \mid n$ , then  $\dim \tilde{J}^{(2)} > 1$ .*

*Proof.* This follows from the remark on page 163 of [Maz77]. Since the proof is only sketched there, we give further details for the convenience of the reader. Because  $4 \mid n$ , the cuspidal subgroup  $C$ , which is generated in  $J_0(p)$  by  $(0) - (\infty)$  and is cyclic of order  $n$ , contains an element of order 4. Let  $C(2)$  be the 2-primary part of  $C$ , and let  $D = \ker(J_0(p) \rightarrow \tilde{J}^{(2)})$ . If there is a nonzero element in the kernel of the homomorphism  $C(2) \rightarrow \tilde{J}^{(2)}$ , then  $D[\mathfrak{m}] \neq 0$ , where  $\mathfrak{m}$  is the 2-Eisenstein prime. But then by Lemma 2.7, there is an  $A_f \subset D$  such that  $A_f[\mathfrak{m}] \neq 0$ . By Lemma 2.5,  $A_f^\vee$  is a quotient of  $\tilde{J}^{(2)}$ , so  $A_f \subset (\tilde{J}^{(2)})^\vee$  so  $A_f$  cannot be in  $D$ . This contradiction shows that the map  $C(2) \rightarrow \tilde{J}^{(2)}$  is injective, so  $\tilde{J}^{(2)}$  contains a rational point of order 4. However, as mentioned in the introduction,  $E_0(\mathbf{Q})$  has order 2, so  $\tilde{J}^{(2)} \neq E_0$ . Thus  $\tilde{J}^{(2)}$  has dimension bigger than 1.  $\square$

Having established the above lemmas, we are now ready to deduce the theorem.

*Proof of Theorem 2.1.* It seems more straightforward to prove the equivalent statement that the modular degree is even if and only if  $u \equiv 7 \pmod{8}$ , so we will prove this instead.

( $\implies$ )  $u \equiv 7 \pmod{8}$  implies that the modular degree is even: Writing  $u = 8k + 7$  we see that  $p = (8k + 7)^2 + 64 \equiv 1 \pmod{16}$ , so  $16 \mid (p - 1)$  hence  $4 \mid n$ . By Lemma 2.3 and Lemma 2.5,  $E_0$  is a factor of  $\tilde{J}^{(2)}$ . By Lemma 2.8, the dimension of  $\tilde{J}^{(2)}$  is bigger than 1, so by Lemma 2.5 there is an  $A_f$  distinct from  $E_0$  such that  $A_f[\mathfrak{m}] \neq 0$ . Since  $A_f \subset B = \ker(J_0(p) \rightarrow E_0)$ , it follows that  $B[\mathfrak{m}] \neq 0$ . As discussed on page 38 of [Maz77],  $J[\mathfrak{m}]$  has dimension 2 over  $\mathbf{F}_2$  so  $E_0[\mathfrak{m}] = J[\mathfrak{m}]$ , hence  $B[\mathfrak{m}] \subset E_0[\mathfrak{m}]$ . It follows that  $2 \mid \#(B \cap E_0)$ , so  $E_0$  has even modular degree.

( $\impliedby$ ) *Modular degree even implies that  $u \equiv 7 \pmod{8}$ :* Suppose that the modular degree  $m$  of  $E_0$  is even. Letting  $B = \ker(J_0(p) \rightarrow E_0)$ , we have

$$E_0 \cap B \cong \ker(E_0 \rightarrow J_0(p) \rightarrow E_0),$$

so  $\Psi := E_0 \cap B = E_0[m]$ . Lemma 2.3 and our assumption that  $m$  is even imply that

$$E_0[\mathfrak{m}] = E_0[2] \subset E_0[m] = \Psi,$$

so  $\Psi[\mathfrak{m}] \neq 0$ . Since  $\Psi[\mathfrak{m}] \neq 0$ , and  $\Psi \subset B$ , we have  $B[\mathfrak{m}] \neq 0$ . By Lemma 2.7, there is some  $A_f \subset B$  such that  $A_f[\mathfrak{m}] \neq 0$ . Then by Lemma 2.5 we see that  $A_f$  is an isogeny factor of  $\tilde{J}^{(2)}$ . Thus  $\tilde{J}^{(2)}$  has dimension bigger than 1. If  $u = 8k + 3$ , then  $p = (8k + 3)^2 + 64 \equiv 9 \pmod{16}$ , so that  $2 \nmid n$ . However, when  $2 \nmid n$ , [Maz77, Prop. III.7.5] implies that  $\tilde{J}^{(2)} = E_0$ , which is false, so  $u \equiv 7 \pmod{8}$ .  $\square$

*Remark 2.9.* Frank Calegari observed that Lemma 2.8 and its converse also follow from conditions (i) and (v) of Théorème 3 of [Mer96].

### 3. THE STEVENS CONJECTURE FOR NEUMANN–SETZER CURVES IS TRUE

Let  $E$  be an arbitrary elliptic curve over  $\mathbf{Q}$  of conductor  $N$ . Stevens conjectured in [Ste89] that the optimal quotient of  $X_1(N)$  in the isogeny class of  $E$  is the curve in the isogeny class of  $E$  with minimal Faltings height. In this section we explain why this conjecture is true when  $N$  is prime.

Let  $p = u^2 + 64$  be prime and  $E_1$  and  $E_0$  be as in Section 1. In this section we verify that the curve  $E_1$  has smaller Faltings height than  $E_0$ , then show that  $E_1$  is  $X_1(p)$ -optimal. The Stevens conjecture asserts that the  $X_1(p)$ -optimal curve is the curve of minimal Faltings height in an isogeny class, so our results verify the conjecture for Neumann–Setzer curves. In fact, the Stevens conjecture is true for all isogeny classes of elliptic curves of prime conductor. For if  $E$  is an elliptic curve of prime conductor, then by [Set75] there is only one curve in the isogeny class of  $E$ , unless  $E$  is a Neumann–Setzer curve or the conductor of  $E$  is 11, 17, 19, or 37. When the isogeny class of  $E$  contains only one curve, that curve is obviously both  $X_1$ -optimal and of minimal Faltings height. The conjecture is also well-known to be true for curves of conductor 11, 17, 19, or 37 (see [Ste89]). We note that Vatsal [Vat03] has recently extended results of Tang [Tan97] that make considerable progress toward the Stevens conjecture, but his work is not applicable to Neumann–Setzer curves.

**Lemma 3.1.** *The curve  $E_1$  has smaller Faltings height than  $E_0$ .*

*Proof.* By [Ste89, Thm. 2.3, pg. 84] it is enough to exhibit an isogeny from  $E_1$  to  $E_0$  whose extension to Néron models is étale. Let  $\varphi$  be the isogeny  $E_0 \rightarrow E_1$  of degree 2 whose kernel is the subgroup generated by the point whose coordinates are  $(u/4, -u/8)$  in terms of the Weierstrass equation (1.1) for  $E_0$ , which is a global minimal model for  $E_0$ . The kernel of  $\varphi$  does not extend to an étale group scheme over  $\mathbf{Z}$ , since its special fiber at 2 is not étale (it has only one  $\overline{\mathbf{F}}_2$ -point), so the morphism on Néron models induced by  $E_0 \rightarrow E_1$  cannot be étale, since kernels of étale morphisms are étale. By [Ste89, Lemma 2.5] the dual isogeny  $E_1 \rightarrow E_0$  extends to an étale morphism of Néron models.  $\square$

**Proposition 3.2.** *The curve  $E_1$  is  $X_1(p)$ -optimal.*

*Proof.* By [MO89, §5, Lem. 3],  $E_0$  is an optimal quotient of  $X_0(p)$ , so we have an injection  $E_0 \hookrightarrow J_0(p)$ . As in [Maz77, pg. 100], let  $\Sigma$  be the kernel of the functorial map  $J_0(p) \rightarrow J_1(p)$  induced by the cover  $X_1(p) \rightarrow X_0(p)$ . By [Maz77, Prop. II.11.6],  $\Sigma$  is the Cartier dual of the constant subgroup scheme  $U$ , which turns out to equal  $J_0(p)(\mathbf{Q})_{\text{tor}}$ . Because  $\#(E_0 \cap U) = 2$  and  $E_0[2]$  is self dual, we have  $\#(E_0 \cap \Sigma) = 2$ . Thus the image of  $E_0$  in  $J_1(p)$  is the quotient of  $E_0$  by the subgroup generated by the rational point of order 2 (note that the Cartier dual of  $\mathbf{Z}/2\mathbf{Z}$  is  $\mu_2 = \mathbf{Z}/2\mathbf{Z}$ ). This quotient is  $E_1$ , so  $E_1 \subset J_1(p)$ , which implies that  $E_1$  is an optimal quotient of  $X_1(p)$ , as claimed.  $\square$

*Remark 3.3.* The above proposition could also be proved in a slightly different manner. The Faltings height of an elliptic curve is  $\sqrt{2\pi/\Omega}$  where  $\Omega$  is the volume of the fundamental parallelogram associated to the curve. When the conductor is prime, we have by [AL96] that the Manin constants for  $X_0(p)$  and  $X_1(p)$  are 1; this says that for a  $G$ -optimal curve  $E$ , the period lattice generated by  $G$  has covolume equal to  $\Omega_E$ . Since the lattice generated by  $\Gamma_1(p)$  is contained in the lattice generated by  $\Gamma_0(p)$  (and thus has larger covolume), the Faltings height of the  $X_1(p)$ -optimal curve must be less than or equal to that of the  $X_0(p)$ -optimal curve. So if these two curves differ, the  $X_1(p)$ -optimal curve must have smaller Faltings height.

*Remark 3.4.* On page 12 of [Maz98], there is a “To be removed from the final draft” comment that asks (in our notation) whether  $E_0$  is  $X_0(p)$ -optimal when  $p \equiv 1 \pmod{16}$ . This is already answered by [MO89], whereas here we go further and show additionally that  $E_1$  is  $X_1(p)$ -optimal.

## 4. CONJECTURES

**4.1. Refinement of Theorem 2.1.** The following conjectural refinement of Theorem 2.1 is supported by the experimental data of [Wat02]. It is unclear whether the method of proof of Theorem 2.1 can be extended to prove this conjecture.

**Conjecture 4.1.** *If  $u \equiv 7 \pmod{8}$ , then 2 exactly divides the modular degree of  $E_0$  if and only if  $u \equiv 7 \pmod{16}$ .*

We can note that the pattern seems to end here; for curves with  $u \equiv 15 \pmod{16}$  the data give no further information about the 2-valuation of the modular degree. For instance, with  $u = -17$  we have that  $[1, 1, 1, -2, 16]$  has modular degree  $2^3 \cdot 3$ , while with  $u = 175$  the curve  $[1, 1, 1, -634, -6484]$  has modular degree  $2^2 \cdot 3^3 \cdot 5 \cdot 23$ . Similarly, we have that  $u = -33$  gives the curve  $[1, -1, 1, -19, 68]$  with modular degree  $2^5 \cdot 3$ , while  $u = 127$  gives the curve  $[1, 1, 1, -332, -2594]$  of modular degree  $2^2 \cdot 3^2 \cdot 5 \cdot 43$ .

**4.2. The Parity of the Modular Degree.** According to Cremona's tables [Cre], of the 29755 new optimal elliptic curve quotients of  $J_0(N)$  with  $N < 8000$ , a mere 89 have odd modular degree, which is less than 0.3%. There are 52878 non Neumann–Setzer curves in the database of [BM90] with prime conductor  $N \leq 10^7$ ; of these curves 4592, or 8%, have odd modular degree (see [Wat02]). One reason that curves tend to have even modular degree is that for many curves the modular parametrization factors through an Atkin–Lehner quotient. Note that the method of [Wat02] used to compute the modular degree is rigorous when the level is prime because by [AL96] the Manin constant is 1 when the level is odd and square-free.

If  $f(x) = (8x + 3)^2 + 64$ , then it is a well-known conjecture (see [HL22] and e.g., [Guy94, §A1]) that there are infinitely many primes of the form  $f(n)$  for some integer  $n$ , thus we make the following conjecture.

**Conjecture 4.2.** *There are infinitely many elliptic curves over  $\mathbf{Q}$  with odd modular degree.*

Our data suggest the following conjecture:

**Conjecture 4.3.** *If  $E$  is an optimal elliptic curve quotient of  $J_0(p)$  with  $p \not\equiv 3 \pmod{8}$  and  $E$  is not a Neumann–Setzer curve then the modular degree of  $E$  is even or  $p = 17$ .*

There are 23442 Brumer–McGuinness (see [BM90]) curves of conductor  $37 \leq p \leq 10^7$  with  $p \equiv 3 \pmod{8}$ , of which 11815 have even functional equation, of which 7322 have rank 0, and 4589 have odd modular degree. The significance of the data concerning the rank is that the second author has conjectured that  $2^r$  divides the modular degree, where  $r$  is the rank.

*Remark 4.4.* Instead of asking about divisibility by 2, one could ask about divisibility by  $p$ . The first author and Frank Calegari make a conjecture about discriminants of Hecke algebras in [CS03] that implies that the modular degree of an elliptic curve of prime conductor  $p$  is not divisible by  $p$ . This conjecture agrees with our data.

**4.3. Shafarevich–Tate Groups of Neumann–Setzer Curves.** We consider the distribution of III in the Neumann–Setzer family (and note that similar phenomena occur in the related families listed in [SW02]). We look at  $u$  with  $u^2 + 64$  prime and less than  $2 \cdot 10^{12}$ . We now take  $u$  to be positive, which thus replaces the restriction that  $u$  be 3 mod 4. The heuristics of [Del01] would seem to give us an idea of how

TABLE 1. Frequency of a prime dividing III

restriction	number	$p = 3$	$p = 5$	$p = 7$	$p = 11$
$u \equiv 1 \pmod{8}$	25559	33.2%	16.9%	9.2%	3.0%
$u \equiv 3 \pmod{8}$	25557	39.7%	20.3%	14.3%	8.4%
$u \equiv 5 \pmod{8}$	25584	36.2%	18.5%	11.5%	5.0%
$u \equiv 7 \pmod{8}$	25612	34.3%	20.3%	14.3%	8.2%
$u \equiv 0 \pmod{3}$	34009	36.0%	18.7%	12.1%	6.0%
$u \equiv 1 \pmod{3}$	34032	35.2%	18.6%	11.5%	5.6%
$u \equiv 2 \pmod{3}$	34271	36.3%	19.7%	13.3%	6.9%
$u \equiv 0 \pmod{5}$	34208	33.1%	18.0%	11.4%	5.4%
$u \equiv 2 \pmod{5}$	33879	37.1%	19.5%	12.8%	6.5%
$u \equiv 3 \pmod{5}$	34225	37.3%	19.5%	12.7%	6.5%
total	102312	35.8%	19.0%	12.3%	6.2%
Delaunay		36.1%	20.7%	14.5%	9.2%

often we expect a given prime to divide III. For instance, since Neumann–Setzer curves have rank 0, the prime 3 should divide III about 36.1% of the time. However, Table 1 gives a slightly different story with effects seen that depend on the various congruential properties of  $u$ .

## REFERENCES

- [AL96] A. Abbes, E. Ullmo, *À propos de la conjecture de Manin pour les courbes elliptiques modulaires* (French). *Compositio Math.* **103** (1996), no. 3, 269–286.
- [BM90] A. Brumer, O. McGuinness, *The behavior of the Mordell–Weil group of elliptic curves*. *Bull. Amer. Math. Soc. (N.S.)* **23** (1990), no. 2, 375–382, data available online at <http://modular.fas.harvard.edu/~oisin>
- [CS03] F. Calegari, W. Stein, *Conjectures About Discriminants of Hecke Algebras of Prime Level*, submitted (2003).
- [Cre] J. E. Cremona, *Elliptic Curves of conductor  $\leq 17000$* , electronic tables available online at <http://www.maths.nott.ac.uk/personal/jec/ftp/data>
- [Del01] C. Delaunay, *Heuristics on Tate–Shafarevitch Groups of Elliptic Curves Defined over  $\mathbf{Q}$* . *Experiment. Math.* **10** (2001), no. 2, 191–196.
- [Del83] P. Deligne, *Preuve des conjectures de Tate et de Shafarevitch (d’après G. Faltings)*. (French). *Seminaire Bourbaki*, Vol. 1983/84. Astérisque No. 121–122, (1985), 25–41.
- [DI95] F. Diamond and J. Im, *Modular forms and modular curves*, *Seminar on Fermat’s Last Theorem*, Providence, RI, 1995, pp. 39–133.
- [Eme01] M. Emerton, *Optimal Quotients of Modular Jacobians*, preprint (2001).
- [Frey87] G. Frey, *Links between solutions of  $A - B = C$  and elliptic curves*. In *Number Theory (Ulm, 1987)*, edited by H. P. Schlickewei and E. Wirsing, 31–62, *Lecture Notes in Mathematics*, 1380, Springer-Verlag, New York, 1989.
- [Guy94] R. K. Guy, *Unsolved problems in number theory*, Springer-Verlag, 1994.
- [HL22] G. H. Hardy, J. E. Littlewood, *Some Problems of ‘Partitio Numerorum.’ III. On the Expression of a Number as a Sum of Primes*. *Acta. Math.* **44** (1922), 1–70.
- [LO91] S. Ling, J. Oesterlé, *The Shimura subgroup of  $J_0(N)$* . *Astérisque* **196–197** (1991), 171–203.
- [Maz77] B. Mazur, *Modular curves and the Eisenstein ideal*. *Inst. Hautes Études Sci. Publ. Math.* **47** (1977), 33–186.
- [Maz98] B. Mazur, *Three Lectures about the Arithmetic of Elliptic Curves*. Rough, unedited, and preliminary notes from lectures given at the 1998 Arizona Winter School. Available at <http://swc.math.arizona.edu/notes/files/98MazurLN.ps>
- [Mer96] L. Merel, *L’accouplement de Weil entre le sous-groupe de Shimura et le sous-groupe cuspidal de  $J_0(p)$* , *J. Reine Angew. Math.* **477** (1996), 71–115.
- [MO89] J.-F. Mestre, J. Oesterlé, *Courbes de Weil semi-stables de discriminant une puissance  $m$ -ième* (French). *J. Reine Angew. Math.* **400** (1989), 173–184.

- [Mum70] D. Mumford, *Abelian varieties*, Published for the Tata Institute of Fundamental Research, Bombay, 1970, Tata Institute of Fundamental Research Studies in Mathematics, No. 5.
- [Neu71] O. Neumann, *Elliptische Kurven mit vorgeschriebenem Reduktionsverhalten. I, II* (German). Math. Nachr. **49** (1971), 107–123, **56** (1973), 269–280.
- [Ogg74] A. Ogg, *Hyperelliptic modular curves*. Bull. Soc. Math. France **102** (1974), 449–462.
- [RS01] K. A. Ribet and W. A. Stein, *Lectures on Serre’s conjectures*, Arithmetic algebraic geometry (Park City, UT, 1999), IAS/Park City Math. Ser., vol. 9, Amer. Math. Soc., Providence, RI, 2001, pp. 143–232.
- [Set75] B. Setzer, *Elliptic Curves of prime conductor*. J. London Math. Soc. (2), **10** (1975), 367–378.
- [Ste89] G. Stevens, *Stickelberger elements and modular parametrizations of elliptic curves*. Invent. Math. **98** (1989), no. 1, 75–106.
- [SW02] W. A. Stein, M. Watkins, *A Database of Elliptic Curves—First Report*. In *Algorithmic number theory* (Sydney 2002), 267–275, edited by C. Fieker and D. Kohel, Lecture Notes in Comput. Sci., 2369, Springer, Berlin, 2002.
- [Tan97] S.-L. Tang, *Congruences between modular forms, cyclic isogenies of modular elliptic curves, and integrality of  $p$ -adic  $L$ -function*. Trans. Amer. Math. Soc. **349** (1997), no. 2, 837–856.
- [Vat03] V. Vatsal, *Multiplicative subgroups of  $J_0(N)$  and applications to elliptic curves*, preprint (2003).
- [Wat02] M. Watkins, *Computing the modular degree of an elliptic curve*, Experiment. Math. **11** (2002), no. 4, 487–502.
- [Wil95] A. J. Wiles, *Modular elliptic curves and Fermat’s last theorem*. Ann. of Math. (2) **141** (1995), no. 3, 443–551.