Appendix by A. Agashe and W. Stein.

In this appendix, we apply a result of J. Sturm* to obtain a bound on the number of Hecke operators needed to generate the Hecke algebra as an abelian group. This bound was suggested to the authors of this appendix by Loïc Merel and Ken Ribet.

Theorem. *The ring* $\mathbf{T}$ *of Hecke operators acting on the space of cusp forms of weight* k *and level* N *is generated as an abelian group by the Hecke operators* T_n *with*

$$n \leq \frac{kN}{12} \cdot \prod_{p\mid N} \left(1 + \frac{1}{p}\right).$$

Proof. For any ring R, let $S_k(N;R) = S_k(N;\mathbf{Z}) \otimes R$, where $S_k(N;\mathbf{Z})$ is the subgroup of cusp forms with integer Fourier expansion at the cusp ∞, and let $\mathbf{T}_R = \mathbf{T} \otimes_{\mathbf{Z}} R$. There is a perfect pairing $S_k(N;R) \otimes_R \mathbf{T}_R \to R$ given by $\langle f, T\rangle \mapsto a_1(T(f))$.

Let M be the submodule of $\mathbf{T}$ generated by $T_1, T_2, \ldots, T_r$, where r is the largest integer $\leq \frac{kN}{12} \cdot \prod_{p\mid N}\left(1 + \frac{1}{p}\right)$. Consider the exact sequence of additive abelian groups

$$0 \to M \xrightarrow{i} \mathbf{T} \to \mathbf{T}/M \to 0.$$

Let p be a prime and use that tensor product is right exact to obtain an exact sequence

$$M \otimes \mathbf{F}_p \xrightarrow{\bar{i}} \mathbf{T} \otimes \mathbf{F}_p \to (\mathbf{T}/M) \otimes \mathbf{F}_p \to 0.$$

Suppose that $f \in S_k(N;\mathbf{F}_p)$ pairs to 0 with each of $T_1, \ldots, T_r$. Then $a_m(f) = a_1(T_m f) = \langle f, T_m\rangle = 0$ in $\mathbf{F}_p$ for each $m \leq r$. By Theorem 1 of Sturm's paper, it follows that $f = 0$. Thus the pairing restricted to the image of $M \otimes \mathbf{F}_p$ in $\mathbf{T} \otimes \mathbf{F}_p$ is nondegenerate, so

$$\dim_{\mathbf{F}_p} \bar{i}(M \otimes \mathbf{F}_p) = \dim_{\mathbf{F}_p} S_k(N, \mathbf{F}_p) = \dim_{\mathbf{F}_p} \mathbf{T} \otimes \mathbf{F}_p.$$

It follows that $(\mathbf{T}/M) \otimes \mathbf{F}_p = 0$; repeating the argument for all primes p shows that $\mathbf{T}/M = 0$, as claimed.

Remark. In general, the theorem is not true if one considers only T_n where n runs over the *primes* less than the bound. Consider, for example, $S_2(11)$, where the bound is 2 and T_2 is the 1×1 matrix [2], which does not generate the full Hecke algebra as a $\mathbf{Z}$-submodule of $\mathrm{End}(S_2(\Gamma_0(N), \mathbf{Z}))$. One needs, in addition, the matrix [1].

* J. Sturm, *On the Congruence of Modular Forms*. Number theory (New York, 1984–1985), 275–280, Lecture Notes in Math., 1240, Springer, Berlin-New York, 1987.