Modular Symbols, Manin Symbols and Modular Forms

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Outline

Modular Forms are Computable

Recall that the space of classical weight k level N **modular forms** is

$$M_k(\Gamma_1(N)) = \left\{ f : \mathfrak{h} \to \mathbb{C} \text{ such that } f^{[\gamma]_k} = f \text{ all } \gamma \in \Gamma_1(N), \text{etc.} \right\}$$

Theorem

The space $M_k(\Gamma_1(N))$ is computable*, i.e., there is an algorithm that takes as input k, N, B and outputs a basis of q-expansions for $M_k(\Gamma_1(N))$ to precision $O(q^B)$.*

This theorem is not at all obvious from the definitions!

Modular Forms are Computable: Example

```
sage: M = ModularForms(Gamma1(2006),2)
sage: M
Modular Forms space of dimension 127136 for Congruence Subgroup
Gamma1(2006) of weight 2 over Rational Field
```

OK, that's *not* computing... But this is!

```
sage: M = ModularForms(Gamma1(13),2,prec=13)
sage: M.basis()
[
q - 4*q^3 - q^4 + 3*q^5 + 6*q^6 - 3*q^8 + q^9 - 6*q^10 - 2*q^12 + O(q^
q^2 - 2*q^3 - q^4 + 2*q^5 + 2*q^6 - 2*q^8 + q^9 - 3*q^10 + O(q^13),
1 + 21060/19*q^11 - 36504/19*q^12 + O(q^13),
q + 11709/19*q^11 - 20687/19*q^12 + O(q^13),
q^2 + 262*q^11 - 467*q^12 + O(q^13),
q^3 + 918/19*q^11 - 1215/19*q^12 + O(q^13),
q^4 - 882/19*q^11 + 2095/19*q^12 + O(q^13),
q^5 - 1287/19*q^11 + 2607/19*q^12 + O(q^13),
q^6 - 1080/19*q^11 + 2024/19*q^12 + O(q^13),
q^7 - 675/19*q^11 + 1056/19*q^12 + O(q^13),
q^8 - 360/19*q^11 + 453/19*q^12 + O(q^13),
q^9 - 153/19*q^11 + 98/19*q^12 + O(q^13),
q^10 - 54/19*q^11 - 9/19*q^12 + O(q^13)
]
```

Weight 2 Modular Symbols

Definition (Weight 2 Modular Symbols)

The group $\mathcal{M}_2$ is the free abelian group on symbols $\{\alpha, \beta\}$ with

$$\alpha, \beta \in \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}$$

modulo the relations

$$\{\alpha, \beta\} + \{\beta, \gamma\} + \{\gamma, \alpha\} = 0,$$

for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$, and all torsion.

I.e.,

$$\mathcal{M}_2 = (F/R)/(F/R)_{\text{tor}},$$

where F is the free abelian group on all **ordered** pairs (α, β) and R is the subgroup generated by all elements of the form $(\alpha, \beta) + (\beta, \gamma) + (\gamma, \alpha)$.

Remark

$\mathcal{M}_2$ is a **HUGE** free abelian group of countable rank.

Remark

$\mathcal{M}_2$ is the **relative homology** of the upper half plane relative to the cusps.

Weight k Modular Symbols

For any integer $n \geq 0$, let $\mathbb{Z}[X, Y]_n$ be the abelian group of homogeneous polynomials of degree n in two variables X, Y.

Remark

$\mathbb{Z}[X, Y]_n$ is isomorphic to $\mathrm{Sym}^n(\mathbb{Z} \times \mathbb{Z})$ as a group, but certain actions are different...

Fix an integer $k \geq 2$.

Definition (Modular Symbols of Weight k)

Set

$$\boldsymbol{\mathcal{M}}_k = \mathbb{Z}[X, Y]_{k-2} \otimes_{\mathbb{Z}} \boldsymbol{\mathcal{M}}_2,$$

which is a torsion-free abelian group whose elements are sums of expressions of the form $X^i Y^{k-2-i} \otimes \{\alpha, \beta\}$.

Example

$$X^3 \otimes \{0, 1/2\} - 17XY^2 \otimes \{\infty, 1/7\} \in \boldsymbol{\mathcal{M}}_5.$$

Some Representations of $\mathrm{SL}_2(\mathbb{Z})$

Fix: **finite index** subgroup

$$\Gamma \subset \mathrm{SL}_2(\mathbb{Z}).$$

Definition

Define a **left action of Γ** on $\mathbb{Z}[X,Y]_{k-2}$ as follows. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and $P(X,Y) \in \mathbb{Z}[X,Y]_{k-2}$, let

$$(gP)(X,Y) = P(dX - bY, -cX + aY).$$

Remark

If we think of $z = (X,Y)$ as a column vector, then

$$(gP)(z) = P(g^{-1}z),$$

since $g^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$.

The **reason for the inverse** is so that this is a left action instead of a right action, e.g., if $g, h \in \Gamma$, then

$$((gh)P)(z) = P((gh)^{-1}z) = P(h^{-1}g^{-1}z) = (hP)(g^{-1}z) = (g(hP))(z).$$

Weight k Level Γ Modular Symbols

Let Γ act on the left on $\boldsymbol{\mathcal{M}}_2$ by

$$g\{\alpha,\beta\} = \{g(\alpha), g(\beta)\}.$$

Here Γ acts via linear fractional transformations, so if $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, then

$$g(\alpha) = \frac{a\alpha + b}{c\alpha + d}.$$

Combine these two actions to obtain a left action of Γ on $\boldsymbol{\mathcal{M}}_k$, which is given by

$$g(P \otimes \{\alpha,\beta\}) = (gP) \otimes \{g(\alpha), g(\beta)\}.$$

For example,

$$\begin{aligned}\begin{pmatrix} 1 & 2 \\ -2 & -3 \end{pmatrix} . (X^3 \otimes \{0, 1/2\}) &= (-3X - 2Y)^3 \otimes \left\{-\frac{2}{3}, -\frac{5}{8}\right\} \\ &= (-27X^3 - 54X^2Y - 36XY^2 - 8Y^3) \otimes \left\{-\frac{2}{3}, -\frac{5}{8}\right\}.\end{aligned}$$

We will often write $P(X,Y)\{\alpha,\beta\}$ for $P(X,Y) \otimes \{\alpha,\beta\}$.

Definition (Modular Symbols of Weight k and Level Γ)

Let $k \geq 2$ be an integer and let Γ be a finite index subgroup of $\mathrm{SL}_2(\mathbb{Z})$. The space $\boldsymbol{\mathcal{M}}_k(\Gamma)$ of weight k modular symbols for Γ is the quotient of $\boldsymbol{\mathcal{M}}_k$ by all relations $gx - x$ for $x \in \boldsymbol{\mathcal{M}}_k$, $g \in \Gamma$, and by any torsion.

Example: Weight k Level Γ Modular Symbols

```
sage: ModularSymbols(Gamma1(13),2)
Modular Symbols space of dimension 15 for Gamma_1(13) of weight 2 with
sign 0 and over Rational Field

sage: ModularSymbols(Gamma0(1),24)
Modular Symbols space of dimension 5 for Gamma_0(1) of weight 24 with
sign 0 over Rational Field

sage: ModularSymbols(Gamma0(11),2)
Modular Symbols space of dimension 3 for Gamma_0(11) of weight 2 with
sign 0 over Rational Field

sage: set_modsym_print_mode('modular')
sage: M = ModularSymbols(Gamma0(1),24)
sage: M.basis()
(X^18*Y^4*{0,Infinity}, X^19*Y^3*{0,Infinity}, X^20*Y^2*{0,Infinity},
```

That's weird – the dimensions are twice as big as you might expect... but actually that's correct.

Manin Symbols

$$\mathcal{M}_k(\Gamma) = \text{quotient of two infinite rank abelian groups}$$

Theorem (Manin, Shokurov)

$\mathcal{M}_k(\Gamma)$ is computable.

Suppose $P \in \mathbb{Z}[X, Y]_{k-2}$ and $g \in \mathrm{SL}_2(\mathbb{Z})$. Then the **Manin symbol** associated to this pair of elements is

$$[P, g] = g(P\{0, \infty\}) \in \mathcal{M}_k(\Gamma).$$

Proposition

If $\Gamma g = \Gamma h$, then $[P, g] = [P, h]$.

Proof.

The symbol $g(P\{0, \infty\})$ is invariant by the action of Γ on the left, since it is a modular symbols for Γ. □

For a right coset Γg we also write $[P, \Gamma g]$ for the symbol $[P, h]$ for any $h \in \Gamma g$.

Manin Symbols Generate

The abelian group generated by Manin symbols is of finite rank, generated by

$$\{[X^{k-2-i}Y^i, \Gamma g_j] : i = 0, \ldots, k-2, \quad \text{and} \quad j = 0, \ldots, r\},$$

where $g_0, \ldots, g_r$ run through representatives for the right cosets $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$.

Proposition

The Manin symbols generate $\boldsymbol{\mathcal{M}}_k(\Gamma)$.

Proof.

We just do the case $k = 2$. Suffices to prove for $\{0, b/a\}$, where b/a is in lowest terms. Induct on $a \in \mathbb{Z}_{\geq 0}$. Assume $a > 0$ (case $a = 0$ trivial). Find $a' \in \mathbb{Z}$ with

$$ba' \equiv 1 \pmod{a}.$$

Then $b' = (ba' - 1)/a \in \mathbb{Z}$, so $\delta = \begin{pmatrix} b & b' \\ a & a' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Thus $\delta = \gamma \cdot g_j$ for some right coset representative g_j and $\gamma \in \Gamma$. Then

$$\{0, b/a\} - \{0, b'/a'\} = \{b'/a', b/a\} = \begin{pmatrix} b & b' \\ a & a' \end{pmatrix} \cdot \{0, \infty\} = g_j\{0, \infty\}.$$

□

$SL_2(\mathbb{Z})$-Action on Manin Symbols

Let

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix}, \qquad J = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Define a **right action of $SL_2(\mathbb{Z})$** on Manin symbols as follows. If $h \in SL_2(\mathbb{Z})$, let

$$[P, g]h = [h^{-1}P, gh].$$

This is a right action because $P \mapsto h^{-1}P$ is, as is right multiplication $g \mapsto gh$.

Manin Symbols Presentation

Theorem (Manin)

Form the free abelian group F on formal symbols

$$\{[X^{k-2-i}Y^i, \Gamma g_j]' \ : \ i = 0, \dots, k-2, \quad \text{and} \quad j = 0, \dots, r\},$$

where $g_0, \dots, g_r$ run through representatives for the right cosets $\Gamma \backslash \mathrm{SL}_2(\mathbb{Z})$. Let V be the quotient of F by the subgroup generated by all

$$z + z\sigma \qquad \text{and} \qquad z + z\tau + z\tau^2,$$

where $z = [X^{k-2-i}Y^i, \Gamma g_j]'$, and modulo any torsion. Then there is an isomorphism

$$V \xrightarrow{\sim} \mathcal{M}_k(\Gamma)$$

given by

$$[X^{k-2-i}Y^i, \Gamma g_j]' \mapsto [X^{k-2-i}Y^i, \Gamma g_j].$$

In other words, computing $\mathcal{M}_k(\Gamma)$ is straightforward sparse linear algebra!!!
(We proved surjectivity. Gabor might prove injectivity...)

Example: Manin Symbols Presentation

```
sage: set_modsym_print_mode('manin')        # back to default.
sage: M = ModularSymbols(Gamma0(11),2)
sage: S = M.manin_symbols()
sage: S.manin_symbol_list()
[(0,1), (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6),
 (1,7), (1,8), (1,9), (1,10)]
sage: import sage.modular.modsym.relation_matrix as r
sage: r.relation_matrix_wtk_g0(S,0,QQ,4)[0]     # after quotient by 2-t
[ 0  0  0  0  0  0  0  0  0  0  0  1]
[ 0  0  0  0  0  0  0  0  0  0  0  1]
[ 0  0  0  0  0  0  1  0  0  0  1 -1]
[ 0  0  0  0  0  0 -1  0  1 -1  0  0]
[ 0  0  0  0  0  0  0  0 -1  1 -1  0]
[ 0  0  0  0  0  0 -1  0  1 -1  0  0]
[ 0  0  0  0  0  0  1  0  0  0  1 -1]
[ 0  0  0  0  0  0  0  0 -1  1 -1  0]
[ 0  0  0  0  0  0 -1  0  1 -1  0  0]
[ 0  0  0  0  0  0  0  0 -1  1 -1  0]
[ 0  0  0  0  0  0  1  0  0  0  1 -1]
[ 0  0  0  0  0  0  0  0  0  0  0  1]
sage: M.basis()
((1,0), (1,8), (1,9))
```

Pairing Modular Symbols and Modular Forms

Let

$$\overline{S}_k(\Gamma) = \{\bar{f} : f \in S_k(\Gamma)\}.$$

Define a pairing

$$(S_k(\Gamma) \oplus \overline{S}_k(\Gamma)) \times \mathcal{M}_k(\Gamma) \to \mathbb{C} \tag{3.1}$$

by letting

$$\langle (f_1, f_2), P\{\alpha, \beta\}\rangle = \int_\alpha^\beta f_1(z)P(z,1)\,\mathrm{d}z + \int_\alpha^\beta f_2(z)P(\overline{z},1)d\overline{z},$$

and extending linearly.

Proposition

The integration pairing is well defined, i.e., if we replace $P\{\alpha, \beta\}$ by an equivalent modular symbols (equivalent modulo the left action of Γ), then the integral is the same.

Remark

Modular symbols are constructed exactly so the integration pairing is well defined!

Special Values of L-functions

Modular symbols were introduced by Bryan Birch in the 1960s in order to study special values of L-functions and the BSD conjecture.
The L-function of a cusp form $f = \sum a_n q^n \in S_k(\Gamma_1(N))$ is

$$L(f,s) = (2\pi)^s \Gamma(s)^{-1} \int_0^{\infty} f(it) t^s \frac{dt}{t} \tag{3.2}$$

$$= \sum_{n=1}^{\infty} \frac{a_n}{n^s} \qquad \text{for } \mathrm{Re}(s) \gg 0. \tag{3.3}$$

For each integer j with $1 \leq j \leq k-1$, we have setting $s = j$ and making the change of variables $t \mapsto -it$ in (3.2), that

$$L(f,j) = \frac{(-2\pi i)^j}{(j-1)!} \cdot \left\langle f,\ X^{j-1} Y^{k-2-(j-1)} \{0,\infty\} \right\rangle . \tag{3.4}$$

Remark

Neither the pairing nor computation of $L(f,j)$ is implemented in SAGE yet, though I implemented both in Magma.

Cuspidal and Boundary Modular Symbols

Let $\mathcal{B}$ be the free abelian group on symbols $\{\alpha\}$, for $\alpha \in \mathbb{P}^1(\mathbb{Q})$, and set

$$\mathcal{B}_k = \mathbb{Z}[X, Y]_{k-2} \otimes \mathcal{B}.$$

Left action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathcal{B}_k$:

$$g.(P\{\alpha\}) = (gP)\{g(\alpha)\},$$

for $g \in \mathrm{SL}_2(\mathbb{Z})$. Let $\mathcal{B}_k(\Gamma)$ be the quotient of $\mathcal{B}_k$ by the relations $x - g.x$ for all $g \in \Gamma$ and by any torsion. Thus $\mathcal{B}_k(\Gamma)$ is a torsion free abelian group.

Definition

The **boundary map** is the map $b : \mathcal{M}_k(\Gamma) \to \mathcal{B}_k(\Gamma)$ given by extending the map $b(P\{\alpha, \beta\}) = P\{\beta\} - P\{\alpha\}$ linearly.

Definition

The space $\mathcal{S}_k(\Gamma)$ of **cuspidal modular symbols** is the kernel

$$\mathcal{S}_k(\Gamma) = \ker(\mathcal{M}_k(\Gamma) \to \mathcal{B}_k(\Gamma)).$$

We have an exact sequence

$$0 \to \mathcal{S}_k(\Gamma) \to \mathcal{M}_k(\Gamma) \to \mathcal{B}_k(\Gamma),$$

which is exact on the right if $k > 2$.

Pairing With Modular Forms

Theorem (Shokurov)

The pairing

$$\langle \cdot\, , \cdot \rangle : (S_k(\Gamma) \oplus \overline{S}_k(\Gamma)) \times \boldsymbol{\mathcal{S}}_k(\Gamma, \mathbb{C}) \to \mathbb{C}$$

is a nondegenerate pairing of complex vector spaces.

Theorem

If

$$f = (f_1, f_2) \in S_k(\Gamma_1(N)) \oplus \overline{S}_k(\Gamma_1(N))$$

and $x \in \boldsymbol{\mathcal{M}}_k(\Gamma_1(N))$, *then for any* n,

$$\langle T_n(f), x \rangle = \langle f, T_n(x) \rangle.$$

The Star Involution

On Manin symbols, ι^* it is

$$\iota^*[P,(u,v)] = -[P(-X,Y),(-u,v)]. \tag{3.5}$$

(Here (u,v) are the bottom two entries of a matrix, which uniquely determines a right coset of $\Gamma_1(N)$.)

Let $\boldsymbol{\mathcal{S}}_k(\Gamma)^+$ be the $+1$ eigenspace for ι^* on $\boldsymbol{\mathcal{S}}_k(\Gamma)$, and let $\boldsymbol{\mathcal{S}}_k(\Gamma)^-$ be the -1 eigenspace. There is also a map ι on modular forms, which is adjoint to ι^*.

Theorem

The integration pairing $\langle\cdot\,,\,\cdot\rangle$ induces nondegenerate Hecke compatible bilinear pairings

$$\boldsymbol{\mathcal{S}}_k(\Gamma)^+ \times S_k(\Gamma) \to \mathbb{C} \qquad \textit{and} \qquad \boldsymbol{\mathcal{S}}_k(\Gamma)^- \times \overline{S}_k(\Gamma) \to \mathbb{C},$$

so

$$\boldsymbol{\mathcal{S}}_k(\Gamma)^+ \approx S_k(\Gamma)$$

as modules over the Hecke algebra.

Thus computing modular symbols allows one to compute modular forms.

Example: Star Involution

```
sage: M = ModularSymbols(Gamma0(11),2)

sage: M.star_involution()
Hecke module morphism Star involution on Modular Symbols space of
dimension 3 for Gamma_0(11) of weight 2 with sign 0 over Rational
Field defined by the matrix
[ 1  0  0]
[ 0 -1  1]
[ 0  0  1]
Domain: Modular Symbols space of dimension 3 for Gamma_0(11)
   of weight ...
Codomain: Modular Symbols space of dimension 3 for Gamma_0(11)
   of weight ...

sage: ModularSymbols(Gamma0(11),2,sign=1)
Modular Symbols space of dimension 2 for Gamma_0(11) of weight 2 with
sign 1 over Rational Field

sage: ModularSymbols(Gamma0(11),2,sign=-1)
Modular Symbols space of dimension 1 for Gamma_0(11) of weight 2 with
sign -1 over Rational Field
```

Summary

1. Modular symbols are a **purely algebraic construction** involving a quotient of two infinitely generated abelian groups.
2. Modular symbols have a **finite easy-to-compute presentation** in terms of Manin symbols.
3. Modular symbols are **dual to modular forms**, hence they allow one to compute modular forms.