Computing with Siegel Modular Forms

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Why one might study Siegel modular forms

- they are multivariate elliptic modular forms
- they can be related to the number of ways of representing a quadratic form by another
- they have many applications: Coding Theory (Choie, Duke), Conformal Field Theory (Tuite), Special Values of L-functions (Fukuda-Komatsu), etc.

What's needed for modular forms

- an upper half-space
- an arithmetic group acting on the upper half-space
- a functional equation and automorphy factor
- a Fourier expansion

An Upper-Half Space

- Let
$$\mathfrak{h}_g = \{Z \in M_g(\mathbb{C}) : Z = {}^tZ, \mathrm{Im}(Z) > 0\}$$
be the **Siegel upper half-space of genus** g.
  - $\mathfrak{h}_1$ = Poincaré upper half plane
  - Since the $Z \in \mathfrak{h}_g$ are symmetric $g \times g$ matrices, we see there are $\frac{g(g+1)}{2}$ free variables.
  - For $g > 1$ the upper half-space is hard to picture. In particular, $\mathfrak{H}_2$ is bounded by 28 algebraic surfaces.

An Arithmetic Group

- Let $\mathrm{Sp}_g(\mathbb{R})$ be

$$\left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2g}(\mathbb{R}) : {}^tBD, {}^tAC \text{ symm. } {}^tAD - {}^tCB = I_g \right\},$$

  the **symplectic group** of size $2g$.
  - $\mathrm{Sp}_1(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})$.
  - $\Gamma_g = \mathrm{Sp}_g(\mathbb{Z})$ is **Siegel's modular group**. The notion of congruence subgroups of Γ_g also translates nicely.

An action

- Let $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_g(\mathbb{R})$. Then

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot Z = (AZ + B)(CZ + D)^{-1}$$

  defines an action on $\mathfrak{h}_g$.
  - For $g = 1$, this corresponds to the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane.
  - We must show that $CZ + D$ is invertible, but that's a straightforward exercise in linear algebra.

Siegel Modular Forms

- Let $\mathcal{M}_k(\Gamma_g) = \mathcal{M}_k^g$ be the space of **Siegel modular forms of weight k and genus g**. I.e., $F \in \mathcal{M}_k^g$ iff
  - $F : \mathfrak{h}_g \longrightarrow \mathbb{C}$ is holomorphic,
  - $F\left((AZ+B)(CZ+D)^{-1}\right) = \det(CZ+D)^k F(Z)$ for all $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_g$
  - $F(Z) = \sum_{T \geq 0} a(T) e^{\pi i \operatorname{tr}(TZ)}$ where T runs over all positive semi-definite even integral $g \times g$ matrices.
- We remark that
  - the existence of a Fourier expansion like the one above is a theorem for $g \geq 2$
  - if the Fourier expansion is supported only on positive definite forms $F \in \mathcal{S}_k^g$, i.e., is a **cusp form**.

An Example

- Let S be an even unimodular matrix of size m (by a result of Hecke we know such a thing exists iff $8|m$). Then for $Z \in \mathfrak{h}_g$, define

$$\Theta_S^{(g)}(Z) = \sum_{N \in M_{m \times g}} e^{\pi i \mathrm{tr}({}^t NSNZ)}.$$

  - if $r(S, T)$ is the number of ways of respresenting T by S, then

$$\Theta_S^{(g)}(Z) = \sum_{N \geq 0} r(S, N) e^{\pi i \mathrm{tr}(TZ)}$$

  - $\Theta_S^{(g)}(Z) \in M_{m/2}^g$.

Hecke operators

- Let $\Gamma := \Gamma_g$ and $G = \mathrm{GSp}_g^+(\mathbb{Q})$ be the group of **rational symplectic similitudes with positive scalar factor**.
- Let $L(\Gamma, G)$ be the free $\mathbb{C}$-module generated by the right cosets $\Gamma\alpha$ where $\alpha \in \Gamma\backslash G$.
- Γ acts on $L(\Gamma, G)$ be right multiplication and we set

$$\mathcal{H}_g(\Gamma, G) = L(\Gamma, G)^{\Gamma}.$$

Hecke operators form an algebra

- Let $T_1, T_2 \in \mathcal{H}_g(\Gamma, G)$ and
$$T_i = \sum_{\alpha_i \in \Gamma \backslash G} c_i(\alpha)\Gamma\alpha.$$
Then
$$T_1 T_2 = \sum_{\alpha,\alpha' \in \Gamma \backslash G} c_1(\alpha)c_2(\alpha')\Gamma\alpha\alpha'.$$

Local Hecke algebras

- $\mathcal{H}_g = \bigotimes_{p \text{ prime}} \mathcal{H}_{g,p}$ where the construction of the **local Hecke algebra** $\mathcal{H}_{g,p}$ is the same as before but with G replaced with $G_p = G \cap \mathrm{GL}_{2g}(\mathbb{Z}[p^{-1}])$.
- $\mathcal{H}_{g,p}$ is generated by the double cosets

$$T(p) = \Gamma \mathrm{diag}(I_g; pI_g)\Gamma \text{ and}$$
$$T_i(p^2) = \Gamma \mathrm{diag}(I_i, pI_{g-i}; p^2 I_i, pI_{g-i})\Gamma.$$

Punchline: Knowing the action of the generators in principle means knowing the algebra.

Slash operator

- $\mathcal{H}_g$ acts on $\mathcal{M}_k^g$ by

$$F|_k \left(\sum c_i \Gamma \alpha_i \right) = \sum c_i F|_k \alpha_i$$

where

$$(F|_k \alpha)(Z) = r(\alpha)^{gk - \frac{g(g+1)}{2}} \det(CZ + D)^{-k} F(\alpha \cdot Z)$$

Satake isomorphism

- In the 1960s Satake proved the following theorem (in much more generality):

$$\mathcal{H}_{g,p} \cong \mathbb{C}[x_0^{\pm 1}, \ldots, x_g^{\pm 1}]^{W_g}$$

where W_g is the Weyl group generated by the permutations of $x_1, \ldots, x_g$ and by the maps $x_0 \mapsto x_0 x_j$, $x_j \mapsto x_j^{-1}$, $x_i \mapsto x_j$ $(i \neq j,\ 1 \leq i \leq g)$.

Call this map the **spherical map** and denote it by Ω.

Satake parameters

- What Satake really proved (again in more generality) was:

$$\mathrm{Hom}_{\mathbb{C}}\left(\mathcal{H}_{g,p}, \mathbb{C}\right) = \left(\mathbb{C}^{\times}\right)^{g+1} / W_g.$$

  Let Ψ denote the isomorphism.

- Let F be an eigenform for all the Hecke operators and for $T \in \mathcal{H}_g$ write $F|_k T = \lambda_F(T)F$. Then

$$\Psi(T \mapsto \lambda_F(T)) = (\alpha_{0,p}, \ldots, \alpha_{g,p}).$$

  The entries of the above $(g+1)$-tuple are the **Satake parameters** of F.

Lifts

- Let f be an (elliptic) simultaneous eigencuspform of weight $2k$. I.e., $f \in \mathcal{S}^g_{2k}$. Ikeda (2001) showed that (roughly) there exists a form $F \in \mathcal{S}^{2g}_{k+g}$ (if k, g have the same parity) so that the L-functions of f and F (almost) coincide. In particular,

$$L^{\mathrm{std}}(F,s) = \zeta(s) \prod_{i=1}^{2g} L(f, s+k+g-i).$$

  where
  - $L^{\mathrm{std}}(F,s) = \prod_p [L_p(F,s)(p^{-s})]^{-1}$ where

$$L_p(F,X) = \prod_{i=1}^{g} (1-\alpha_{i,p}X)(1-\alpha_{i,p}^{-1}X).$$

  In the literature, this is called the **Ikeda** lift.
- Ikeda (2006) also proved the existence of a lift from $S^1_{2k} \otimes S^r_{k+r+n} \to S^{2n+r}_{k+r+n}$. In the literature this is called the **Miyawaki** lift.

How I tackle problems for Siegel modular forms

- What's the statement/solution to the problem for genus 1 modular forms?
- What's the statement/solution to the problem for lifts?
- What's the statement/solution for nonlifts?

The problems I've been involved in:

- (1) How can I compute the Satake parameters of a Siegel modular form?
- (2) What kinds of complex numbers are the Satake parameters?
- (3) Is there a Maeda-type conjecture for Siegel modular forms of genus 2?
- (4) How can we compute Siegel modular forms in genus 2?
- (5) What can be said about the L-functions of genus 4 modular forms?

I will try to provide short answers to these which will necessarily be incomplete.

Ramanujan τ-function

- $\tau(n)$ is defined by:

$$(2\pi)^{-12}\Delta(z) = q\prod_{n\geq 1}(1-q^n)^{24} =_{\text{def}} \sum_{n\geq 1}\tau(n)q^n$$

- the L-function associate to Δ can be thought to have denominator at p of:

$$1-\tau(p)X+p^{11}X^2$$

- Ramanujan conjectured the roots of this polynomial were complex conjugate

Satake parameters of elliptic forms

- Let $f \in \mathcal{S}_k^1$ be a simultaneous eigenform, and let $T(p)|_k f = \lambda_p f$. Then $(\beta_{0,p}, \beta_{1,p})$ is the solution to

$$\beta_0^2 \beta_1 = p^{k-1}$$
$$\beta_0(1 + \beta_1) = \lambda_p$$

- $\beta_0\beta_1$ and β_0 are roots of

$$1 - a(p)X + p^{k-1}X^2$$

where $a(p)$ is the pth eigenvalue of f

Satake parameters of Ikeda lifts ...

- The parameters of a lift (to genus 2) are given by:

$$\alpha_{0,p} = p^{k-2}$$
$$\alpha_{1,p} = p^{2-k}\beta_{0,p}$$
$$\alpha_{2,p} = p^{2-k}\beta_{0,p}\beta_{1,p}$$

- Similar formulas hold for arbitrary g

Key result to compute with

Theorem (R. 2006)

Grade the Hecke algebra. Then, the matrix representation of Ω *restricted to a direct summand is square and upper triangular. Moreover, the entries of the matrix are computed explicitly.*

- It appears that this result might have some application to the study of buildings.

Satake parameters of nonlifts in genus 2

- In genus 2 the parameters are the roots of the following polynomial:

$$P_4(x) = x^4 - c_1 x^3 + (c_2 + 2)x^2 - c_1 x + 1$$

where c_1 and c_2 are explicit constants that depend on p and the form F.

Computing in Arbitrary Genus

Theorem (Poor, Yuen, R. 2006)

Let $F \in \mathcal{S}_k(\Gamma_g)$ be a simultaneous eigenform. Given the eigenvalues of F with respect to the generators $T(p)$, $T_1(p^2)$, $\ldots$, $T_g(p^2)$ of the local Hecke algebra, we construct a polynomial whose roots are the Satake p-parameters of F.

Degrees of Satake parameters

We have the following bounds for the degrees:

- For **lifts**:

$$\deg \alpha_0 = 1 \quad \deg \alpha_1 = \deg \alpha_2 = d \text{ or } 2d$$

  where $d = \dim \mathcal{S}_k^1$.

- For **nonlifts**:

$$\deg \alpha_0 \leq 4D^5 \quad \deg \alpha_1 = \deg \alpha_2 \leq 4D^2$$

  (where D is the dimension of $\mathcal{S}_k^?(\Gamma_2)$, the space of nonlifts).

Maeda-type conjecture

- It is known that

$$\mathcal{S}_k(\Gamma_2) \equiv \mathcal{S}_{2k-2} \oplus \mathcal{S}_k^?$$

  is a Hecke invariant splitting and so the characteristic polynomials of the $T(p)$ is reducible, so the obvious Maeda conjecture does not hold.
- Skoruppa showed that even on $S_{24}^?(\Gamma_2)$ the characteristic polynomial splits (into linear terms).

Maeda-type conjecture

- Computational evidence suggests that for **lifts**

$$\deg(\alpha_0) = 1 \text{ and } \deg(\alpha_1) = \deg(\alpha_2) = 2\dim(\mathcal{S}_k^1)$$

while for **nonlifts**

$$\deg(\alpha_1) = \deg(\alpha_2) \leq 4\dim(\mathcal{S}_k^?(\Gamma_2)$$

- We can interpret this as "Satake parameters are as irreducible as possible," like Maeda's conjecture says that "Hecke eigenvalues are as irreducible as possible".

Computations in genus 2

This conjecture has been verified up to weight 48, in joint work with David Yuen. Our method is as follows:

- $\mathcal{M}_*^2 = \mathbb{C}[E_4, E_6, X_{10}, Y_{12}]$
- Determine these forms as theta lifts
- Find a basis of S_k^2 in terms of these forms
- Find the action of $T(p)$ and $T(p^2)$ on S_k^2
- Compute Satake parameters

Computations in genus 4

- Poor and Yuen have determined the spaces of genus 4 in weights 8 through 16.
- Poor, Yuen and I have (almost) identified which forms can be Ikeda lifts, which are Miyawaki lifts, and the last form in weight 16 appears to be neither. Is this a new lift?