Lecture 4: Examples of automorphic forms on the unitary group U(3)

Lassina Dembélé
Department of Mathematics
University of Calgary

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Motivation

The main goal of this talk is to show how one can compute automorphic forms on the unitary group in three variables.

Notations

- F is the field of rationals or a real quadratic field and E is a totally CM quadratic extension of F.
- The involution in $\mathrm{Gal}(E/F)$ is denoted by $a \mapsto \bar{a}$, $a \in E$.
- The rings of integers of F and E by $\mathcal{O}_F$ and $\mathcal{O}_E$, respectively.
- For any prime $\mathfrak{p}$ in $\mathcal{O}_F$, we denote by $F_\mathfrak{p}$ and $\mathcal{O}_{F,\mathfrak{p}}$ the completions of F and $\mathcal{O}_F$ at $\mathfrak{p}$, respectively.

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Unitary groups

For any F-algebra A, the $\mathrm{Gal}(E/F)$-action induces an involution of the matrix group $\mathbf{GL}_3(E \otimes_F A)$ we denote as before.

The unitary group in three variables $\mathbf{U}(3)$ on F attached to E is defined as follows. For any F-algebra A, the set of A-rational points on $\mathbf{U}(3)/F$ is given by

$$\mathbf{U}(3)(A) = \left\{ g \in \mathbf{GL}_3(A \otimes_F E) : \ g\bar{g}^t = \mathbf{1}_3 \right\} .$$

Unitary groups

The unitary group in three variables $\mathbf{U}(2,1)$ on F attached to E is defined as follows. For any F-algebra A, the set of A-rational points on $\mathbf{U}(2,1)/F$ is given by

$$\mathbf{U}(2,1)(A) = \left\{ g \in \mathbf{GL}_3(A \otimes_F E) : \; g \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bar{g}^t = \mathbf{1}_3 \right\}.$$

Unitary groups

We define an integral structure on $\mathbf{U}(3)/F$ which we denote the same way by putting

$$\mathbf{U}(3)(A) = \{g \in \mathbf{GL}_3(A \otimes_{\mathcal{O}_F} \mathcal{O}_E) : \; g\bar{g}^t = \mathbf{1}_3, \; \in A^\times\},$$

for any $\mathcal{O}_F$-algebra A.

Unitary groups

We define an integral structure on $\mathbf{U}(2,1)/F$ which we denote the same way by putting

$$\mathbf{U}(2,1)(\mathcal{O}_E \otimes_{\mathcal{O}_F} A) = \left\{ g \in \mathbf{GL}_3(A \otimes_F E) : \ g \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \bar{g}^t = \mathbf{1}_3 \right\},$$

for any $\mathcal{O}_F$-algebra A.

$\mathbf{U}(2,1)$ versus SL_2

SL_2	$\mathrm{U}(2,1)$
Global symmetric space: $\mathfrak{H}$, the Poincaré upper-half plane	Global symmetric space: $B = \{(z,u) \in \mathbb{C}^2 : 2Re(z) + \|u\|^2 < 0\}$
Congruence subgroups: $\Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$	Congruence subroups: $\Gamma_0(N) \subseteq \mathbf{U}(2,1)(\mathbb{Z})$
$\Gamma_0(N)\backslash\mathfrak{H}^*$, compact arithmetic curves	$\Gamma_0(N)\backslash B^*$, compact arithmetic surfaces

U(2, 1) versus SL_2

SL_2	$\mathrm{U}(2,1)$
Inner forms: quaternions algebras	Inner forms: **U**(3) and (certain) division algebras, with involution of the second kind.
Jaquet-Langlands	Jacquet-Langlands.

Goal

We want to construct automorphic forms on $\mathbf{U}(2,1)$ by using its inner form $\mathbf{U}(3)$.

Study the Galois representations we obtain from those automorphic forms.

Assumption: Assume throughout this paper that F has narrow class number 1 and that the quadratic extension E/F is chosen so that the associated group $\mathbf{U}(3)/F$ has class number 1.

Unitary groups: local case

- Let $\mathfrak{p}$ be a prime in F and choose a prime $\mathfrak{P}$ of E above $\mathfrak{p}$.
- Then, $K_\mathfrak{p} = \mathbf{U}(3)(\mathcal{O}_{F,\mathfrak{p}})$ is a maximal compact open subgroup in $\mathbf{U}(3)(F_\mathfrak{p})$.
- When $\mathfrak{p}$ is split in E, we choose an isomorphism

$$\mathbf{U}(3)(F_\mathfrak{p}) \cong \mathbf{GL}_3(E_\mathfrak{P}) = \mathbf{GL}_3(F_\mathfrak{p}) \text{ s.t. } K_\mathfrak{p} = \mathbf{U}(3)(\mathcal{O}_\mathfrak{p}) \cong \mathbf{GL}_3(\mathcal{O}_\mathfrak{p}).$$

- When $\mathfrak{p}$ is inert in E, then $\mathbf{U}(3)/F_\mathfrak{p}$ is the unique unitary group in three variables on $F_\mathfrak{p}$ attached to the quadratic extension $E_\mathfrak{P}/F_\mathfrak{p}$, and $K_\mathfrak{p} = \mathbf{U}(3)(\mathcal{O}_\mathfrak{p})$ is hyperspecial.

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Automorphic forms

- We let K be the product $K = \prod_{\mathfrak{p}} K_{\mathfrak{p}}$, and fix a compact open subgroup U of K.
- Let V be an irreducible algebraic representation of $\mathbf{U}(3)$ defined over F.

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Automorphic forms

Definition

The space of automorphic forms of level U and weight V on $\mathbf{U}(3)$ *is given by*

$$\mathcal{A}_V(U) = \{f : \mathbf{U}(3)(\mathbb{A}_f)/U \to V : f\|\gamma = f,\ \gamma \in \mathbf{U}(3)(F)\},$$

where $f\|\gamma(x) = f(\gamma x)\gamma$ *for all* $x \in \mathbf{U}(3)(\mathbb{A}_f)$ *and* $\gamma \in \mathbf{U}(3)(F)$.

Hecke operators

For any $u \in \mathbf{U}(3)(\mathbb{A}_f)$, write $UuU = \coprod_i u_i U$. Define the Hecke operator

$$\begin{aligned}[][UuU] : \mathcal{A}_V(U) &\rightarrow \mathcal{A}_V(U) \\ f &\mapsto f\|[UuU]\end{aligned}$$

by $f\|[UuU](x) = \sum_i f(xu_i), \;\; x \in \mathbf{U}(3)(\mathbb{A}_f)$.

Hecke operators

In the rest of this section, we fix an integral ideal N of F such that $(N, \mathrm{disc}(E/F)) = (1)$, and define the level

$$U_0(N) = \left\{ \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \in K \text{ such that } a_3 \equiv b_3 \equiv 0 \mod N \right\},$$

and to simplify notations, we let $\mathcal{A}_V(N) = \mathcal{A}_V(U_0(N))$ and $\mathbf{T}_V(N) = \mathbf{T}_V(U_0(N))$ be the Hecke algebra.

Hecke operators

Let $\mathfrak{p}$ be a split prime in F and choose a prime $\mathfrak{P}$ in E above $\mathfrak{p}$. Then, the local algebra of $\mathbf{T}_V(N)$ at $\mathfrak{p}$ is isomorphic to the Hecke algebra of $\mathbf{GL}_3(F_\mathfrak{p})$ which is generated by the two operators

$$T_1(\mathfrak{p}) = \left[\mathbf{GL}_3(\mathcal{O}_{F,\mathfrak{p}})\mathrm{diag}(1,\, 1,\, \varpi_\mathfrak{p})\mathbf{GL}_3(\mathcal{O}_{F,\mathfrak{p}})\right] = [\Delta_1(\mathfrak{p})]$$

and

$$T_2(\mathfrak{p}) = \left[\mathbf{GL}_3(\mathcal{O}_{F,\mathfrak{p}})\mathrm{diag}(1,\, \varpi_\mathfrak{p},\, \varpi_\mathfrak{p})\mathbf{GL}_3(\mathcal{O}_{F,\mathfrak{p}})\right] = [\Delta_2(\mathfrak{p})],$$

where $\varpi_\mathfrak{p}$ is a uniformizer at $\mathfrak{p}$.

Main result

Define the two sets

$$\Theta_i(\mathfrak{p}) = \mathbf{U}(3)(\mathcal{O}_F)\backslash\{g \in \mathrm{M}_3(\mathcal{O}_E) : g\bar{g}^t = \pi_{\mathfrak{p}}\mathbf{1}_3 \text{ and } g \in \Delta_i(\mathfrak{p})\},$$

where $\pi_{\mathfrak{p}}$ is a totally positive generator of $\mathfrak{p}$.

The quotient $\mathbf{U}(3)(\hat{\mathcal{O}}_F)/U_0(N)$ is a flag variety over artinian ring $\mathcal{O}_F/N$, which we denote by $\mathcal{H}_0(N)$.

Main result

Theorem

There is a natural isomorphism of Hecke modules

$$\mathcal{A}_V(N) \cong \{f : \mathcal{H}_0(N) \to V \text{ such that } f\|\gamma = f,\ \gamma \in \Gamma\},$$

where $\Gamma = \mathbf{U}(3)(\mathcal{O}_F)/\mathcal{O}_E^\times$ *and the action of the Hecke operators* $T_1(\mathfrak{p})$ *and* $T_2(\mathfrak{p})$ *on the right hand side is given by*

$$f\|T_i(\mathfrak{p})(x) = \sum_{u\in\Theta_i(\mathfrak{p})} f(ux)u,\ \ x \in \mathcal{H}_0(N).$$

Old and new spaces

Let $\mathfrak{p}$ be a split prime in F such that $\mathfrak{p} \mid N$.

- $\pi : \mathcal{H}_0(N) \to \mathcal{H}_0(N/\mathfrak{p})$ the natural surjection.
- Then, the action of the Hecke operators $T_1(\mathfrak{p})$ and $T_2(\mathfrak{p})$ on $\mathcal{A}_V(N/\mathfrak{p})$ is given by

$$f \| T_i(\mathfrak{p})(x) = \sum_{u \in \Theta_i(\mathfrak{p})} f(ux)u, \ \ x \in \mathcal{H}_0(N),$$

  where the summation is now restricted to the elements whose action is non-degenerate.

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Old and new spaces

There are three degeneracy maps

$$\alpha_i(\mathfrak{p}) : \mathcal{A}_V(N/\mathfrak{p}) \to \mathcal{A}_V(N),\ i = 0, 1, 2,$$

where $\alpha_0(\mathfrak{p}) = \pi^*$ is the pullback map, and $\alpha_i(\mathfrak{p}) = \pi^* \circ T_i(\mathfrak{p})$, $i = 1, 2$, which combine to give

$$\begin{aligned} \iota_{\mathfrak{p}} : \mathcal{A}_V(N/\mathfrak{p})^3 &\to \mathcal{A}_V(N) \\ (f_0, f_1, f_2) &\mapsto \sum_{i=0}^{2} \alpha_i(\mathfrak{p})(f_i). \end{aligned}$$

Old and new spaces

Similarly, when $\mathfrak{p}$ is inert in E, there are two degeneracy maps

$$\alpha_i(\mathfrak{p}) : \mathcal{A}_V(N/\mathfrak{p}) \to \mathcal{A}_V(N),\ i = 0, 2,$$

where $\alpha_0(\mathfrak{p}) = \pi^*$, and $\alpha_2(\mathfrak{p}) = \pi^* \circ T_2(\mathfrak{p})$, and which combine to give

$$\begin{array}{rcl} \iota_{\mathfrak{p}} : \mathcal{A}_V(N/\mathfrak{p})^2 & \to & \mathcal{A}_V(N) \\ (f_0, f_1) & \mapsto & \alpha_0(\mathfrak{p})(f_0) + \alpha_2(\mathfrak{p})(f_1). \end{array}$$

Old and new spaces

Definition

The space of oldforms is obtained as

$$\mathcal{A}_V^{old}(N) := \sum_{\mathfrak{p}|N} \mathrm{im}(\iota_{\mathfrak{p}}),$$

and the space of newforms $\mathcal{A}_V^{new}(N)$ *as its orthogonal complement with respect to any* $\mathbf{U}(3)$*-invariant Hermitian inner product* $(\ ,\ )$ *on* $\mathcal{A}_V(N)$.

Examples

- The unitary groups $\mathbf{U}(3)/\mathbb{Q}$ in three variables attached to the quadratic fields $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$ respectively; and also, on the unitary group $\mathbf{U}(3)/\mathbb{Q}(\sqrt{5})$ attached to the cyclotomic field $\mathbb{Q}(\zeta_5)$.
- For each group, we compute the space $\mathcal{A}_0(N)$ of automorphic forms of trivial weight and level N, where $\mathrm{Norm}(N) \leq 20$ and $(N, \mathrm{disc}(E/F)) = 1$. We provide a table for the dimensions of $\mathcal{A}_0(N)$ and $\mathcal{A}_0^{new}(N)$, and the list of all the automorphic forms whose Hecke eigenvalues are rational or defined over a quadratic field.

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The unitary group $U(3)/\mathbb{Q}$ attached to $\mathbb{Q}(\sqrt{-1})$

N	3	5	7	9	11	13	15	17	19
$\dim \mathcal{A}_0(N)$	2	3	6	11	17	7	16	9	77
$\dim \mathcal{A}_0^{new}(N)$	1	2	5	9	16	6	8	8	76

N	3	5	7	9		
p	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_2)$	$a_1(p, f_3)$
2	-1	$1-2\omega_5$	-1	-1	5	-3
3	-3	$12-4\omega_5$	-4	0	0	0
5	3	$-1+2\omega_5$	$-3-6\omega_{-4}$	3	-3	13
13	15	$7+4\omega_5$	$1-10\omega_{-4}$	15	3	3
17	27	$19-16\omega_5$	$-9-12\omega_{-4}$	27	33	1
29	3	$23+8\omega_5$	$11+8\omega_{-4}$	3	69	-11
37	63	$31+20\omega_5$	$-21-24\omega_{-4}$	63	-33	-33
41	99	$19-12\omega_5$	$-25+44\omega_{-4}$	99	-39	121

The unitary group $\mathrm{U}(3)/\mathbb{Q}$ attached to $\mathbb{Q}(\sqrt{-3})$

N	4	5	7	8	10	11	13	14	16	17	19	20
$\dim \mathcal{A}_0(N)$	2	2	3	4	8	8	5	7	24	26	7	50
$\dim \mathcal{A}_0^{new}(N)$	1	1	2	2	5	7	4	2	20	25	6	40

N	4	5	7	8		10
p	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_2)$	$a_1(p, f_1)$
2	0	4	$2-2\omega_{28}$	0	0	1
3	-4	-2	$1+\omega_{28}$	4	-4	-4
5	54	-5	$26+10\omega_{28}$	22	54	1
7	9	1	ω_{28}	1	9	-3
13	-9	29	$4-\omega_{28}$	23	-9	3
19	45	17	$22+5\omega_{28}$	21	45	-3
31	-15	13	$65-2\omega_{28}$	9	-15	-15
37	63	21	$41-14\omega_{28}$	31	63	27
43	21	59	$3+6\omega_{28}$	125	21	21

The unitary group $U(3)/\mathbb{Q}(\sqrt{5})$ attached to $\mathbb{Q}(\zeta_5)$

N	$(4, 2)$	$(9, 3)$	$(11, 3+\omega_5)$	$(16, 4)$
$\dim \mathcal{A}_0(N)$	2	3	3	14
$\dim \mathcal{A}_0^{new}(N)$	1	2	2	12

N		$(4,2)$	$(9,3)$	$(11,3+\omega_5)$	$(16,4)$	
$\mathbf{N}(\mathfrak{p})$	$\mathfrak{p}$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_1)$	$a_1(p, f_2)$
4	2	-4	$2+12\omega_{24}$	$18-2\omega_{44}$	0	0
5	$2+\omega_5$	4	$2+3\omega_{24}$	$6+2\omega_{44}$	4	-2
11	$3+\omega_5$	3	$15-3\omega_{24}$	$12-\omega_{44}$	3	-11
11	$3+2\omega_5$	3	$15-3\omega_{24}$	$-\omega_{44}$	3	5

Residual Galois representations

- Let f be a newform of level N with eigenvalues in the number field K_f, and let $\mathcal{O}_f$ be the ring of integers of K_f.
- Let $\ell \geq 2$ be a prime and choose a prime λ of K_f that lies above ℓ.
- We denote the completions of K_f and $\mathcal{O}_f$ at λ by $K_{f,\lambda}$ and $\mathcal{O}_{f,\lambda}$ respectively.
- Let $\pi_f = \otimes_{\mathfrak{p}} \pi_{\mathfrak{p}}$ be the automorphic representation attached to f, and let $\pi_f^E = \otimes_{\mathfrak{P}} \pi_{\mathfrak{P}}^E$ be the base change lift of π_f to $\mathbf{GL}(3)/E$.
- We denote the Hecke matrix of $\pi_{\mathfrak{P}}^E$ by $t_{\mathfrak{P}}^E$.

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Theorem (Kottwitz)

There exists a Galois representation

$$\rho_{f,\lambda} : \mathrm{Gal}(\bar{E}/E) \to \mathrm{GL}_3(K_{f,\lambda}),$$

associated to f, such that the characteristic polynomial of $\rho_{f,\lambda}(\mathrm{Frob}_{\mathfrak{P}})$ coincides with the one of $t^E_{\mathfrak{P}}$. The representation $\rho_{f,\lambda}$ is unramified outside $\ell \mathrm{disc}(E/F)N$.

By making an appropriate choice of lattice in $K^3_{f,\lambda}$, one can reduce $\rho_{f,\lambda}$ to get a mod λ representation $\bar{\rho}_{f,\lambda}$. The data we have seem to support the following conjecture.

Problem: Numerically study the image of the Galois representation $\bar{\rho}_{f,\ell}$.

- L. Clozel, M. Harris and R. Taylor; Automorphy for some ℓ-adic lifts of automorphic mod ℓ representations (preprint).
- L. Dembélé, Explicit computations of Hilbert modular forms on $\mathbb{Q}(\sqrt{5})$. *Experimental Mathematics* **4**, vol. 14 (2005) pp. 457-466.
- L. Dembélé, Quaternionic *M*-symbols, Brandt matrices and Hilbert modular forms. To appear in *Math. Comp.*
- Hashimoto, Ki-ichiro; Koseki, Harutaka. Class numbers of positive definite binary and ternary unimodular Hermitian forms. *Tohoku Math. J. (2)* **41** (1989), no. 1, 1–30.
- Hashimoto, Ki-ichiro; Koseki, Harutaka. Class numbers of positive definite binary and ternary unimodular Hermitian forms. *Tohoku Math. J. (2)* **41** (1989), no. 2, 171–216.

Kottwitz, Robert E. Shimura varieties and λ-adic representations. Automorphic forms, Shimura varieties, and *L*-functions, Vol. I (Ann Arbor, MI, 1988), 161–209, Perspect. Math., 10, Academic Press, Boston, MA, 1990.

Lansky, Joshua; Pollack, David. Hecke algebras and automorphic forms. *Compositio Math.* **130** (2002), no. 1, 21–48.

Gan, Wee Teck; Hanke, Jonathan P.; Yu, Jiu-Kang. On an exact mass formula of Shimura. *Duke Math. J.* **107** (2001), no. 1, 103–133.