Warm-Up Exercises

Math 98T

Due: Friday, April 4, 2008

Do at least three of the following problems.

1. **Prove or disprove:** Let $f$ be the polynomial $f(x) = x^2 + x + 41$. Then $f(n)$ is a prime number for every integer $n$.

2. We say that a collection of integers are called **relatively prime** if they share no common prime factors. So $2, 4$ and $8$ are not relatively prime, but $4$ and $5$ are relatively prime. Find three positive integers $a, b,$ and $c$ such that the collection of all three are relatively prime, but no pair among them is relatively prime.

3. Find a string of 15 consecutive integers, none of which is prime.
   **Bonus:** Given any integer $n$, does there exist a string of $n$ consecutive integers, none of which is prime? Why or why not?

4. Let’s say I pick two numbers $a$ and $b$, which may or may not be the same. If I give you the sum of these two numbers, and the product of these two numbers, is that enough information to determine $a$ and $b$? If so, give me a recipe for finding $a$ and $b$. If not, explain why not.

5. (Silverman & Tate, Ex. 1.1.) For a point $P = (x, y)$ in the plane, we say that $P$ is a **rational point** if $x, y \in \mathbb{Q}$. Given a line $L$ defined by $ax + by = c$, we say that $L$ is a **rational line** if $a, b, c \in \mathbb{Q}$.
   
   (a) If $P$ and $Q$ are distinct rational points in the plane, prove that the line connecting them is a rational line.
   
   (b) If $L_1$ and $L_2$ are distinct rational lines in the plane, prove that their intersection is either a rational point or empty.

6. Let $f(x) = ax^2 + bx + c$ be a parabola. Give a condition in terms of $a, b,$ and $c$ to decide whether or not the graph of $f$ crosses the $x$-axis.