Cyber-Enabled Connections Between Pure Mathematics and the Functions of Mathematical Physics and Engineering

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1 The Proposal

Part I of the present CDI is easily summarized: *develop and implement the software needed to allow the mathematical content already present at the NIST-based DLMF site to generate "validated" expressions in the syntax of all known Open and Proprietary "engines," including parsing and validation of the many possibilities for differing definitions of the "special functions" in each of these environments, or their complete absence in others.* Sage has already definitively demonstrated to work across all such software engines, in a smooth and efficient manner that neither asks nor requires facilitation by the engine creators. Once this is done, a fully automatic validation of both the DLMF (itself) and of the definitions and manipulations of the functions by the external engines can be carried out. Actually completing all this for the entire DLMF is more than we are likely to achieve given the resources we are requesting; however, we will lay the foundations and complete a substantial chunck of the work. In parallel with Part I, we will work on Part II, which is far more ambitious: can the pure math engine Sage, with the validated mathematical data base represented by the DLMF, act to parse and simplify expressions based on their actual mathematical content (rather than, for example, simply validate relationships by checking numerical evaluations of different expressions) and generate new ones? The existing symbolic mathematical engines have a limited ability to work abstractly with functions more complex that the elementary functions of Chapter XXX of the DLMF, namely the trig, log, and exponential functions. We envisage extending this power of generation and simplifications to essentially "all of the special functions" and the relationships between them. Should 10% of this goal be achieved it will represent a remarkable advance in symbolic mathematics, albeit within the restricted world of the DLMF; but these DLMF functions are precisely the functions of constant use in applied mathematics, and in the physical and engineering sciences!

2 Virtual Organizations, Complex Built Systems, and from Data to Living Knowledge

Sage is already the center of a growing community of over 100 developers and several thousand users; the release, in 2008, of the DLMF web site will immediately create a community of mathematicians and applications people who for years have relied on the static AMS 55. We envisage "chat-" and "wiki-" DLMF sites. We are thus proposing to take the applied and pure mathematical "data" of the DLMF, and convert that validated knowledge base into living math, thus a cyber enabled transformation from static data to living and expanding knowledge. The DLMF and Sage both represent highly heterogeneous "built" and "complex" environments: we propose to merge these environments, and also to merge and unite their user communities. The structures and techniques entering the Sage environment, will, where appropriate, produce corresponding results within the environment of the DLMF: thus pure math and representations of that math in terms of the functions of the DLMF will become transparent: conversely the mathematical content of novel results obtained via combination or manipulation, or study of, the special functions can guide the intuition of pure mathematicians.

3 A Sample Application

Deep research in pure mathematics frequently involves the same mathematical objects as in mathematical physics and engineering, even though the perspective and motivations are vastly different. The project described in this proposal involves bridging this chasm.

Number theorists frequently consider a cubic equation $y^2 = x^3 + Ax + B$ in x and y, with A and B constant whole numbers (with $27B^2 + 4A^3 \neq 0$).

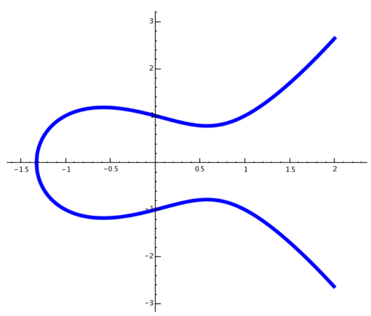

Figure 1: Real Solutions

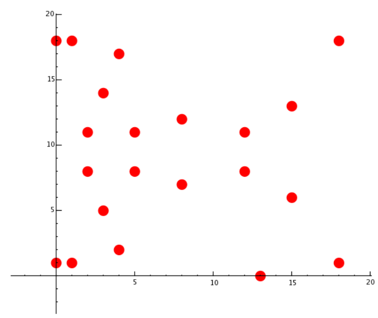

Figure 2: Solutions Modulo 19

For each prime p consider p minus the number of solutions modulo p:

$$a_p = p - \#\{(x, y) : 0 \leq x, y < p \text{ and } p \text{ divides } y^2 - (x^3 + Ax + B)\}.$$

Wiles et al. proved Fermat's Last Theorem by proving that the function $L(s) = \prod_p 1/(1 - a_p p^{-s} + p^{1-2s})$ extends to a holomorphic function on the whole complex plane that satisfies a certain symmetry. The Birch and Swinnerton-Dyer Conjecture (the million dollar Clay Mathematics Millenium Problem in Arithmetic Geometry) asserts that the order of vanishing of $L(s)$ at $s = 1$ is related in a precise way to the group of rational solutions to the equation $y^2 = x^3 + Ax + B$. The Riemann hypothesis (the Clay Millenium problem in analytic number theory) predicts that all zeros of the Riemann Zeta function $\zeta(s)$ of Figure 3 with positive real part have real part $1/2$. In the 19th century, elliptic functions arose out of attempts to understand complex solutions to cubic equations as above; today, explicit computation with $L(s)$ and $\zeta(s)$ plays a major role in number theory, and involves computing with incomplete Gamma functions.

In the physical sciences and engineering, many "special functions" related to the solutions of linear and non-linear ODEs and PDEs, arise in the descriptions of cutting edge physical science, for example, Figure 4 shows a train of bright solitons in a gaseous Bose-Einstein condenstate, an example from physics, whose optical analogs (an engineering application) form the pulses for web communications. The mathematical functions describing these soltions are Jacobian elliptic functions, which are connected, one to another, and to many other special functions by "identities" many of which correspond precisely to group addition laws on the elliptic curve of Figure 1. Yet such identities are, in the eyes of applications minded physical scientists and engineers, analytically obtainable results, found without recourse to their pure math underlying structures. Another example is that the incomplete Gamma fucntions, mentioned above, appear in applications in quantum chemistry, and the dynamics of soft materials. A computer-based bonding of these complementary veiws is long overdue because it is quite a challenge to implement. We accept this challenge.

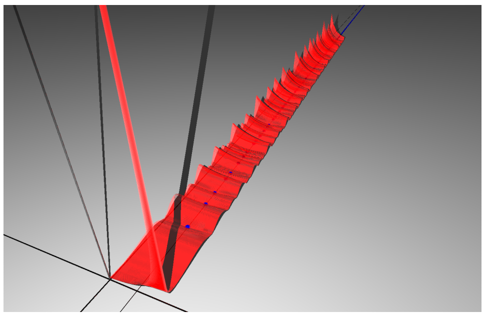

Figure 3: The absolute value of the Riemann Zeta function

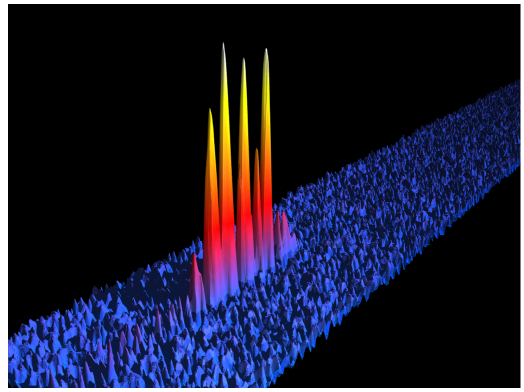

Figure 4: A soliton train in a gaseous Bose-Einstein condensate, described by Jacobian elliptic functions.

We propose to use the open source mathematical software Sage to vastly extend the usability and applicability of the wealth of information encoded in the web based Digital Library of Mathematical Functions (DLMF), and in so doing transform the DLMF into "living" rather than simply "reference" mathematics. This will have a tremendous impact on all sciences that use computation, and forge new connections between pure mathematics and mathematical physics and engineering. In particular, we imagine incorporating directly into Sage as much as possible of the carefully collated citations, references, identities and other painstaking quality work that the DLMF project has created. Because the DLMF is so systematically organized and carefully validated, in many cases incorporating this information into Sage can be automated, and once it is in Sage we can use Sage itself to verify that the claimed identities hold – this will be an additional check on the DLMF itself and whether we correctly represented them in Sage. We envision Sage also automatically making use of identities from the DLMF when simplifying expressions and solving problems. Moreover, as illustrated above the special functions of mathematical physics and engineering have a vast overlap with the special functions of many advanced areas of pure mathematics.

The DLMF is the fully updated, and greatly expanded version of the classic Handbook of Mathematical Fuctions, edited by M. Abramowtiz, and I. Stegun, and originally published (without copyright) but the National Bureau of Standards, NBS, and reprinted by Dover. It is the most referenced "handbook" in all of mathematics. Just as NBS is now NIST, Abramowitz and Stegun is about to appear on the Web as the DLMF, updated and re-authored by over 50 mathematicians for a new age that addresses the needs of users of all stripes.

We propose that Sage and the DLMF together will transform computational mathematics even more than Abramowitz and Stegun has already impacted science and engineering.

4 Sage: Open Source Mathematical Software

The co-PI Stein started the open source software project Sage (`http://sagemath.org`) in 2005 to support advanced research in number theory. Now that a much larger group of people are using and contributing to Sage, the scope of Sage has broadened enormously to include everything from symbolic manipulation to high performance numerical computation.

Sage is the only serious general purpose mathematics software that uses a mainstream programing language as the end user language. The programming language used for working with Sage is Python (`http://python.org`), which is powerful, modern and interpreted. With Python it is easy to define new data types (e.g., bitstreams, ciphers, rings, etc.), and Python has excellent support for bit and string manipulation. Python is a "clean" language that results in readable and maintainable code. Many standard Python libraries are available for statistics, networking, databases, bioinformatics, physics, 3d graphics, cryptography, and numerous other application domains. There is also a Python to C compiler (`http://cython.org`).

Sage is open source, so it is significantly more flexible and extensible than all traditional commercial mathematical software offerings. In particular, everybody is allowed to view and modify absolutely all of the source code of their copy of Sage, change their copy however they want for their needs, and freely share an unlimited number of copies with others. This makes Sage a good long-term investment.

Instead of reinventing the wheel, Sage combines many of the best existing open source libraries that have been developed over the last 40 years (well over 4 million lines of well-tested code) with 200,000 lines of new code. Every copy of Sage includes the following (and much more):

- Commutative algebra: Singular
- Linear algebra: ATLAS, Linbox, and IML
- Graphics: Matplotlib, Tachyon3d, and Java3d
- Group theory: GAP
- High precision arithmetic: GMP, MPFR, MPFI, Quaddouble, and Givaro
- Number theory: PARI and NTL
- Numerical computation and Optimization: Scipy, Numpy and GSL
- Statistics: R
- Symbolic calculus: Maxima

Sage is also able to directly interface with Mathematica, Maple, Maxima, and many other systems, so Sage provides a unique and powerful way to use several different systems together in the same project.

The hard work of over 100 hundred people who have contributed to Sage during the last two years paid off when in November 2007, Sage won first place (3000 euros and a laptop) in the scientific category of the Trophées du Libre, which is a major international free software competition (`http://www.tropheesdulibre.org/?lang=en`). There are over 1000 regular Sage users, and over 5000 people downloaded Sage during the first week of December!

5 DLMF: The Digital Library of Mathematical Functions

In 1964 the National Bureau of Standards, NBS, and now the National Institute for Standards and Technology (NIST), published Volume 55 of its Applied Mathematics Series, AMS 55, edited by M.

Abramowitz and I. Stegun, “Handbook of Mathematical Functions, with Graphs and Tables.” The head of the outside Editorial Board, Phillip Morse (of MIT and Morse and Feshbachs, “Methods of Mathematical Physics,” renown) noted, with clear prescience that with the advent of the (new fangled) “computer” that compilations of the functions (and their values) of mathematics and mathematical physics, the physical sciences, and engineering, would be of greater, rather than lesser, value. This has turned out to have been a remarkable understatement: AMS 55, and the Dover paper Reprint of that “un-copyrighted” NBS document is the most referenced handbook in mathematics, and in the mathematical applications based sciences and engineering, owning an ever increasing “fraction” of citations, in a rapidly increasing literature.

These functions (often called the “special functions” and often related to solutions of the solvable ODEs and PDEs of physics and engineering) are sometimes thought of as fossils from the 18th and 19th centuries, of use only to students and writers of text books, and as static as can be: AMS 55 could be carved in stone, and while always useful, would never need an update! The “truth” is rather different: In 1998 a team of over 50 mathematicians began work on bringing AMS 55 up to date, and as an open and free web document: The Digital Library of Mathematical Functions, or DLMF. The result, about to be released, see `http://dlmf.nist.gov/` for a glimpse and an overview, is 32 fully revised and greatly expanded chapters with three times the number of actual mathematical definitions, identities, and expressions than AMS 55. The need for detailed numerical tabulations having decreased, is easily supplanted by new chapters, new functions, new graphics, often in the full complex plane, new classes of relationships, and new uses of the functions both in pure and applied mathematics. There are now XXX expressions and XXX citations to the primary or applications literature. All chapters have undergone several rounds of validation by independent teams of mathematicians to guarantee an NBS-NIST quality standard: see the website for those involved in this immense effort led by FWJ Olver, Editor in Chief.

What has resulted is a computer accessible compilation: using MathML macros, developed at NIST, as part the underlying source code, all of the functions, and expressions involving them are searchable on the website, references may be directly found and are automatically linked to the definitions of all functions in each expression, even if from different chapters, and all expressions are available as “images” or as “LaTeX code”, which may be downloaded from the website. Thus the DLMF represents the beginnings of a fully cyber-enabled document. DLMF Stage II, representing the initial part of this CDI proposal, will be to allow direct generation of living mathematics, not as part of the DLMF site, but by using Open Math with “content” in addition to “presentation” MathML, and by the proposed interfacing with the Sage environment allow the actual purely mathematical content, already present in the DLMF macros, to automatically parse this content and thereupon turn out, in the appropriate syntax, ASCII expressions which may be easily and fully reliably cut and pasted into mathematical manipulational and computational engines, be it Mathematica, Maxima, Maple, Magma, or Sage itself.