Calculus for Scientists and Engineers: Lecture 1

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1 Basics

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Quick Bio:
   research: number theory, software for math
       UC Berkeley 1995-2000 (Ph.D.)
       Harvard 2000-2005 (Asst. Professor)
       UCSD 2005-?  (tenured assoc. prof)
   hobbies: skateboarding, computers, digital photography

Grading:
   4 quizes, from hw, last half of fri class
   2 midterms, 4th hour, wed 7-7:50pm, feb 1, mar 1;
               only planned use of 4th hour
   1 final, 7pm-10pm on wed mar 22; NOT typical mwf at 4pm time
Grade:
   1. 20% quizes (one dropped), 20% midterm 1, 20% midterm 2, 40% final
   2. 20% quizes (one dropped), 20%midterm 1, 60% final
NO MAKEUP EXAMS
   - 1 double-sided page of notes on exam; NOT on quizes.
   - no calculators on exam or quizes, but ...
             ... I personally love calculators / computers and very
                 strongly encourage their use outside exams ...
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2 The Definite Integral

2.1 The definition of area under curve

Let f be a continuous function on interval $[a, b]$. Divide $[a, b]$ into n subintervals of length $\Delta x = (b - a)/n$. Choose (sample) points x_i^* in ith interval, for each i. The (signed) area between the graph of f and the x axis is approximately

$$\begin{aligned} A_n &\sim f(x_1^*)\Delta x + \cdots + f(x_n^*)\Delta x \\ &= \sum_{i=1}^{n} f(x_i^*)\Delta x. \end{aligned}$$

(The $\sum$ is notation to make it easier to write down and think about the sum.)

Definition 2.1. The (signed) area between the graph of f and the x axis between a and b is

$$\lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right)$$

(Note that $\Delta x = (b-a)/n$ depends on n.)

It is a theorem that the area exists and doesn't depend on the choice of x_i^*.

2.2 Relation between velocity and area

Suppose you're reading a car magazine and there is an article about a new sports car that has this table in it:

Time (seconds)	0	1	2	3	4	5	6
Speed (mph)	0	5	15	25	40	50	60

They claim the car drove 1/8th of a mile after 6 seconds, but this just "feels" wrong... Hmmm... Let's estimate the distance driven using the formula

$$\text{distance} = \text{rate} \times \text{time}.$$

We overestimate by assuming the velocity is a constant equal to the max on each interval:

$$\text{estimate} = 5\cdot 1 + 15\cdot 1 + 25\cdot 1 + 40\cdot 1 + 50\cdot 1 + 60\cdot 1 = \frac{195}{3600}\text{ miles} = 0.054...$$

(Note: there are 3600 seconds in an hour.) But $1/8 \sim 0.125$, so the article is inconsistent. (Doesn't this sort of thing just bug you? By learning calculus you'll be able to double-check things like this much more easily.)

Insight! *The formula for the estimate of distance traveled above looks exactly like an approximation for the area under the graph of the speed of the car!* In fact, if an object has velocity $v(t)$ at time t, then the net change in position from time a to b is

$$\int_a^b v(t)dt.$$

We'll come back to this observation frequently.

2.3 Definition of Integral

Let f be a continuous function on the interval $[a, b]$. The definite integral is just the signed area between the graph of f and the x axis:

Definition 2.2 (Definite Integral).

$$\int_a^b f(x)dx = \lim_{n\to\infty}\left(\sum_{i=1}^{n} f(x_i^*)\Delta x\right),$$

Properties of Integration:

- $\int_a^b f(x)dx = -\int_b^a f(x)dx$
- $\int_a^b c_1 f_1(x) + c_2 f_2(x)dx = c_1\int_a^b f_1(x) + c_2\int_a^b f_2(x)dx.$ (linearity)
- If $f(x) \geq g(x)$ on for all $x \in [a, b]$, then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

There are many other properties.

2.4 The Fundamental Theorem of Calculus

Let f be a continuous function on the interval $[a, b]$. The following theorem is *incredibly* useful in mathematics, physics, biology, etc.

Theorem 2.3. *If $F(x)$ is any differentiable function on $[a, b]$ such that $F'(x) = f(x)$, then*

$$\int_a^b f(x)dx = F(b) - F(a).$$

One reason this is amazing, is because it says that the area under the entire curve is completely determined by the values of a ("magic") auxiliary function *at only* 2 *points*. It's hard to believe. It reduces computing (2.2) to finding a single function F, which one can often do algebraically, in practice.Whether or not one should use this theorem to evaluate an integral depends a lot on the application at hand, of course. One can also use a partial limit via a computer for certain applications (numerical integration).

Example 2.4. I've always wondered exactly what the area is under a "hump" of the graph of sin. Let's figure it out, using $F(x) = -\cos(x)$.

$$\int_0^\pi \sin(x)dx = -\cos(\pi) - (-\cos(0)) = -(-1) - (-1) = 2.$$

But does such an F always exist? The surprising answer is "yes".

Theorem 2.5. *Let $F(x) = \int_a^t f(t)dt$. Then $F'(x) = f(x)$ for all $x \in [a, b]$.*

Note that a "nice formula" for F can be hard to find or even provably non-existent.

The proof of Theorem 2.5 is somewhat complicated but is given in complete detail in Stewart's book, and you should definitely read and understand it.

Sketch of Proof. We use the definition of derivative.

$$\begin{aligned} F'(x) &= \lim_{h\to 0} \frac{F(x+h) - F(x)}{h} \\ &= \lim_{h\to 0} \left(\int_a^{x+h} f(t)dt - \int_a^x f(t)dt \right) /h \\ &= \lim_{h\to 0} \left(\int_x^{x+h} f(t)dt \right) /h \end{aligned}$$

Intuitively, for h sufficiently small f is essentially constant, so $\int_x^{x+h} f(t)dt \sim hf(x)$ (this can be made precise using the extreme value theorem). Thus

$$\lim_{h\to 0} \left(\int_x^{x+h} f(t)dt \right) /h = f(x),$$

which proves the theorem. □