Math 129: Algebraic Number Theory

Lecture 20: Extensions and Normalizations of Valuations

William Stein (based closely on Cassels's *Global Fields* article in Cassels-Fröhlich)

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10 Extensions of Valuations

In this section we continue to tacitly assume that all valuations are nontrivial. We do not assume all our valuations satisfy the triangle

Suppose $K \subset L$ is a finite extension of fields, and that $|\cdot|$ and $\|\cdot\|$ are valuations on K and L, respectively.

Definition 10.1 (Extends). We say that $\|\cdot\|$ *extends* $|\cdot|$ if $|a| = \|a\|$ for all $a \in K$.

Theorem 10.2. *Suppose that K is a field that is complete with respect to $|\cdot|$ and that L is a finite extension of K of degree $N = [L : K]$. Then there is precisely one extension of $|\cdot|$ to K, namely*

$$\|a\| = \left|\mathrm{Norm}_{L/K}(a)\right|^{1/N}, \tag{10.1}$$

where the Nth root is the non-negative real Nth root of the nonnegative real number $\left|\mathrm{Norm}_{L/K}(a)\right|$.

Proof. We may assume that $|\cdot|$ is normalized so as to satisfy the triangle inequality. Otherwise, normalize $|\cdot|$ so that it does, prove the theorem for the normalized valuation $|\cdot|^c$, then raise both sides of (10.1) to the power $1/c$. In the uniqueness proof, by the same argument we may assume that $\|\cdot\|$ also satisfies the triangle inequality.

Uniqueness. View L as a finite-dimensional vector space over K. Then $\|\cdot\|$ is a norm in the sense defined earlier. Hence any two extensions $\|\cdot\|_1$ and $\|\cdot\|_2$ of $|\cdot|$ are equivalent as norms, so induce the same topology on K. But as we have seen, two valuations which induce the same topology are equivalent valuations, i.e., $\|\cdot\|_1 = \|\cdot\|_2^c$, for some positive real c. Finally $c = 1$ since $\|a\|_1 = |a| = \|a\|_2$ for all $a \in K$.

Existence. For a proof of existence in the general case, see refs....

Here we give a proof, which was suggested by Dr. Geyer at the conference out of which the Cassels-Fröhlich book arose. It is valid only when K is locally compact, which is the only case we will use later.

We see at once that the function defined in (10.1) satisfies the condition (i) that $\|a\| \geq 0$ with equality only for $a = 0$, and (ii) $\|ab\| = \|a\| \cdot \|b\|$ for all $a, b \in L$. The difficult part of the proof is to show that there is a constant $C > 0$ such that

$$\|a\| \leq 1 \implies \|1 + a\| \leq C.$$

Note that we do not know (and will not show) that $\|\cdot\|$ as defined by (10.1) is a norm as defined in the previous section, since showing that $\|\cdot\|$ is a norm would entail showing that it satisfies the triangle inequality, which is not obvious.

Choose a basis $b_1, \ldots, b_N$ for L over K. Let $\|\cdot\|_0$ be the max norm on L, so for $a = \sum_{i=1}^N c_i b_i$ with $c_i \in K$ we have

$$\|a\|_0 = \left\| \sum_{i=1}^N c_i b_i \right\|_0 = \max\{|c_i| : i = 1, \ldots, N\}.$$

(Note: in Cassels's original article he let $\|\cdot\|_0$ be *any* norm, but we don't because the rest of the proof does not work, since we can't use homogeneity as he claims to do. This is because it need not be possible to find, for any nonzero $a \in L$ some element $c \in K$ such that $\|ac\|_0 = 1$. This would fail, e.g., if $\|a\|_0 \neq |c|$ for any $c \in K$.) The rest of the argument is very similar to our proof from last time of uniqueness of norms on vector spaces over complete fields.

With respect to the $\|\cdot\|_0$-topology, L has the product topology as a product of copies of K. The function $a \mapsto \|a\|$ is a composition of continuous functions on L with respect to this topology (e.g., $\mathrm{Norm}_{L/K}$ is the determinant, hence polynomial), hence $\|\cdot\|$ defines nonzero continuous function on the compact set

$$S = \{a \in L : \|a\|_0 = 1\}.$$

By compactness, there are real numbers $\delta, \Delta \in \mathbf{R}_{>0}$ such that

$$0 < \delta \leq \|a\| \leq \Delta \qquad \text{for all } a \in S.$$

For any nonzero $a \in L$ there exists $c \in K$ such that $\|a\|_0 = |c|$; to see this take c to be a c_i in the expression $a = \sum_{i=1}^N c_i b_i$ with $|c_i| \geq |c_j|$ for any j. Hence $\|a/c\|_0 = 1$, so $a/c \in S$ and

$$0 \leq \delta \leq \frac{\|a/c\|}{\|a/c\|_0} \leq \Delta.$$

Then by homogeneity

$$0 \leq \delta \leq \frac{\|a\|}{\|a\|_0} \leq \Delta.$$

Suppose now that $\|a\| \leq 1$. Then $\|a\|_0 \leq \delta^{-1}$, so

$$\begin{aligned} \|1+a\| &\leq \Delta \cdot \|1+a\|_0 \\ &\leq \Delta \cdot (\|1\|_0 + \|a\|_0) \\ &\leq \Delta \cdot \left(\|1\|_0 + \delta^{-1}\right) \\ &= C \quad \text{(say)}, \end{aligned}$$

as required. □

Example 10.3. Consider the extension $\mathbf{C}$ of $\mathbf{R}$ equipped with the archimedean valuation. The unique extension is the ordinary absolute value on $\mathbf{C}$:

$$\|x + iy\| = \left(x^2 + y^2\right)^{1/2}.$$

Example 10.4. Consider the extension $\mathbf{Q}_2(\sqrt{2})$ of $\mathbf{Q}_2$ equipped with the 2-adic absolute value. Since $x^2 - 2$ is irreducible over $\mathbf{Q}_2$ we can do some computations by working in the subfield $\mathbf{Q}(\sqrt{2})$ of $\mathbf{Q}_2(\sqrt{2})$.

```
> K<a> := NumberField(x^2-2);
> K;
Number Field with defining polynomial x^2 - 2 over the Rational Field
> function norm(x) return Sqrt(2^Valuation(Norm(x),2)); end function;
> norm(1+a);
1.00000000000000000000000000000
> norm(1+a+1);
1.41421356237309504880168872420
```

The following shows that the extension $\|\cdot\|$ does *not* satisfy the triangle inequality, since we can't take $C \leq 1$.

```
> z := 3+2*a;
> norm(z);
1.00000000000000000000000000000
> norm(z+1);
2.82842712474619009760337744839
```

This finally clearly motivates using axiom (3) in the definition of valuation instead of the triangle inequality!

Remark 10.5. Geyer's existence proof gives (10.1). But it is perhaps worth noting that in any case (10.1) is a consequence of unique existence, as follows. Suppose L/K is as above. Suppose M is a finite Galois extension of K that contains L. Then by assumption there is a unique extension of $|\cdot|$ to M, which we shall also denote by $\|\cdot\|$. If $\sigma \in \mathrm{Gal}(M/K)$, then

$$\|a\|_\sigma := \|\sigma(a)\|$$

is also an extension of $|\cdot|$ to M, so $\|\cdot\|_\sigma = \|\cdot\|$, i.e.,

$$\|\sigma(a)\| = \|a\| \qquad \text{for all } a \in M.$$

But now

$$\mathrm{Norm}_{L/K}(a) = \sigma_1(a) \cdot \sigma_2(a) \cdots \sigma_N(a)$$

for $a \in K$, where $\sigma_1, \ldots, \sigma_N \in \mathrm{Gal}(M/K)$ extend the embeddings of L into M. Hence

$$\begin{aligned} \left|\mathrm{Norm}_{L/K}(a)\right| &= \left\|\mathrm{Norm}_{L/K}(a)\right\| \\ &= \prod_{1\leq n\leq N} \|\sigma_n(a)\| \\ &= \|a\|^N, \end{aligned}$$

as required.

Corollary 10.6. *Let $w_1, \dots, w_N$ be a basis for L over K. Then there are positive constants c_1 and c_2 such that*

$$c_1 \leq \frac{\left\| \sum_{n=1}^{N} b_n w_n \right\|}{\max\{|b_n| : n = 1, \dots, N\}} \leq c_2$$

for any $b_1, \dots, b_N \in K$ not all 0.

Proof. For $\left|\sum_{n=1}^{N} b_n w_n\right|$ and $\max|b_n|$ are two norms on L considered as a vector space over K.

I don't believe this proof, which I copied from Cassels's article. My problem with it is that the proof of Theorem 10.2 does not give that $C \leq 2$, i.e., that the triangle inequality holds for $\|\cdot\|$. By changing the basis for L/K one can make any nonzero vector $a \in L$ have $\|a\|_0 = 1$, so if we choose a such that $|a|$ is very large, then the Δ in the proof will also be very large. One way to fix the corollary is to only claim that there are positive constants c_1, c_2, c_3, c_4 such that

$$c_1 \leq \frac{\left\| \sum_{n=1}^{N} b_n w_n \right\|^{c_3}}{\max\{|b_n|^{c_4} : n = 1, \dots, N\}} \leq c_2.$$

Then choose c_3, c_4 such that $\|\cdot\|^{c_3}$ and $|\cdot|^{c_4}$ satisfies the triangle inequality, and prove the modified corollary using the proof suggested by Cassels. □

Corollary 10.7. *A finite extension of a completely valued field K is complete with respect to the extended valuation.*

Proof. By the proceeding corollary it has the topology of a finite-dimensional vector space over K. (The problem with the proof of the previous corollary is not an issue, because we can replace the extended valuation by an inequivalent one that satisfies the triangle inequality and induces the same topology.) □

When K is no longer complete under $|\cdot|$ the position is more complicated:

Theorem 10.8. *Let L be a separable extension of K of finite degree $N = [L : K]$. Then there are at most N extensions of a valuation $|\cdot|$ on K to L, say $\|\cdot\|_j$, for $1 \leq j \leq J$. Let K_v be the completion of K with respect to $|\cdot|$, and for each j let L_j be the completion of L with respect to $\|\cdot\|_j$. Then*

$$K_v \otimes_K L \cong \bigoplus_{1 \leq j \leq J} L_j \tag{10.2}$$

algebraically and topologically, where the right hand side is given the product topology.

Proof. We already know that $K_v \otimes_K L$ is of the shape (10.2), where the L_j are finite extensions of K_v. Hence there is a unique extension $|\cdot|_j^*$ of $|\cdot|$ to the L_j, and by

Corollary 10.7 the L_j are complete with respect to the extended valuation. Further, by a previous proof, the ring homomorphisms

$$\lambda_j : L \to K_v \otimes_K L \to L_j$$

are injections. Hence we get an extension $\|\cdot\|_j$ of $|\cdot|$ to L by putting

$$\|b\|_j = |\lambda_j(b)|_j^* .$$

Further, $L \cong \lambda_j(L)$ is dense in L_j with respect to $\|\cdot\|_j$ because $L = K \otimes_K L$ is dense in $K_v \otimes_K L$ (since K is dense in K_v). Hence L_j is exactly the completion of L.

It remains to show that the $\|\cdot\|_j$ are distinct and that they are the only extensions of $|\cdot|$ to L.

Suppose $\|\cdot\|$ is any valuation of L that extends $|\cdot|$. Then $\|\cdot\|$ extends by continuity to a real-valued function on $K_v \otimes_K L$, which we also denote by $\|\cdot\|$. (We are again using that L is dense in $K_v \otimes_K L$.) By continuity we have for all $a, b \in K_v \otimes_K L$,

$$\|ab\| = \|a\| \cdot \|b\|$$

and if C is the constant in axiom (iii) for L and $\|\cdot\|$, then

$$\|a\| \leq 1 \implies \|1 + a\| \leq C.$$

(In Cassels, he inexplicable assume that $C = 1$ at this point in the proof.)

We consider the restriction of $\|\cdot\|$ to one of the L_j. If $\|a\| \neq 0$ for some $a \in L_j$, then $\|a\| = \|b\| \cdot \|ab^{-1}\|$ for every $b \neq 0$ in L_j so $\|b\| \neq 0$. Hence either $\|\cdot\|$ is identically 0 on L_j or it induces a valuation on L_j.

Further, $\|\cdot\|$ cannot induce a valuation on two of the L_j. For

$$(a_1, 0, \ldots, 0) \cdot (0, a_2, 0, \ldots, 0) = (0, 0, 0, \ldots, 0),$$

so for any $a_1 \in L_1$, $a_2 \in L_2$,

$$\|a_1\| \cdot \|a_2\| = 0.$$

Hence $\|\cdot\|$ induces a valuation in precisely one of the L_j, and it extends the given valuation $|\cdot|$ of K_v. Hence $\|\cdot\| = \|\cdot\|_j$ for precisely one j.

It remains only to show that (10.2) is a topological homomorphism. For

$$(b_1, \ldots, b_J) \in L_1 \oplus \cdots \oplus L_J$$

put

$$\|(b_1, \ldots, b_J)\|_0 = \max_{1 \leq j \leq J} \|b_j\|_j .$$

Then $\|\cdot\|_0$ is a norm on the right hand side of (10.2), considered as a vector space over K_v and it induces the product topology. On the other hand, any two norms are equivalent, since K_v is complete, so $\|\cdot\|_0$ induces the tensor product topology on the left hand side of (10.2). $\square$

Corollary 10.9. *Suppose $L = K(a)$, and let $f(x) \in K[x]$ be the minimal polynomial of a. Suppose that*

$$f(x) = \prod_{1 \leq j \leq J} g_j(x)$$

in $K_v[x]$, where the g_j are irreducible. Then $L_j = K_v(b_j)$, where b_j is a root of g_j.

11 Extensions of Normalized Valuations

Let K be a complete field with valuation $|\cdot|$. We consider the following three cases:

(1) $|\cdot|$ is discrete non-archimedean and the residue class field is finite.

(2i) The completion of K with respect to $|\cdot|$ is $\mathbf{R}$.

(2ii) The completion of K with respect to $|\cdot|$ is $\mathbf{C}$.

(Alternatively, these cases can be subsumed by the hypothesis that the completion of K is locally compact.)

In case (1) we defined the normalized valuation to be the one such that if Haar measure of the ring of integers $\mathcal{O}$ is 1, then $\mu(a\mathcal{O}) = |a|$. In case (2i) we say that $|\cdot|$ is normalized if it is the ordinary absolute value, and in (2ii) if it is the *square* of the ordinary absolute value:

$$|x+iy| = x^2 + y^2 \qquad \text{(normalized)}.$$

In every case, for every $a \in K$, the map

$$a : x \mapsto ax$$

on K^+ multiplies any choice of Haar measure by $|a|$, and this characterizes the normalized valuations among equivalent ones.

We have already verified the above characterization for non-archimedean valuations, and it is clear for the ordinary absolute value on $\mathbf{R}$, so it remains to verify it for $\mathbf{C}$. The additive group $\mathbf{C}^+$ is topologically isomorphic to $\mathbf{R}^+ \oplus \mathbf{R}^+$, so a choice of Haar measure of $\mathbf{C}^+$ is the usual area measure on the Euclidean plane. Multiplication by $x+iy \in \mathbf{C}$ is the same as rotation followed by scaling by a factor of $\sqrt{x^2+y^2}$, so if we rescale a region by a factor of $x+iy$, the area of the region changes by a factor of the square of $\sqrt{x^2+y^2}$. This explains why the normalized valuation on $\mathbf{C}$ is the square of the usual absolute value. Note that the normalized valuation on $\mathbf{C}$ does not satisfy the triangle inequality:

$$|1+(1+i)| = |2+i| = 2^2+1^2 = 5 \not\leq 3 = 1^2 + (1^2+1^2) = |1| + |1+i|\,.$$

The constant C in axiom (3) of a valuation for the ordinary absolute value on $\mathbf{C}$ is 2, so the constant for the normalized valuation $|\cdot|$ is $C \leq 4$:

$$|x+iy| \leq 1 \implies |x+iy+1| \leq 4.$$

Note that $x^2+y^2 \leq 1$ implies

$$(x+1)^2 + y^2 = x^2 + 2x + 1 + y^2 \leq 1 + 2x + 1 \leq 4$$

since $x \leq 1$.

Lemma 11.1. *Suppose K is a field that is complete with respect to a normalized valuation $|\cdot|$ and let L be a finite extension of K of degree $N = [L : K]$. Then the normalized valuation $\|\cdot\|$ on L which is equivalent to the unique extension of $|\cdot|$ to L is given by the formula*

$$\|a\| = \left|\mathrm{Norm}_{L/K}(a)\right| \qquad \textit{all } a \in L. \tag{11.1}$$

Proof. Let $\|\cdot\|$ be the normalized valuation on L that extends $|\cdot|$. Our goal is to identify $\|\cdot\|$, and in particular to show that it is given by (11.1).

By the preceding section there is a positive real number c such that for all $a \in L$ we have

$$\|a\| = \left|\mathrm{Norm}_{L/K}(a)\right|^c.$$

Thus all we have to do is prove that $c = 1$. In case 2 the only nontrivial situation is $L = \mathbf{C}$ and $K = \mathbf{R}$, in which case $\left|\mathrm{Norm}_{\mathbf{C}/\mathbf{R}}(x+iy)\right| = \left|x^2+y^2\right|$, which is the normalized valuation on $\mathbf{C}$ defined above.

One can argue in a unified way in all cases as follows. Let $w_1, \ldots, w_N$ be a basis for L/K. Then the map

$$\varphi : L^+ \to \bigoplus_{n=1}^{N} K^+, \qquad \sum a_n w_n \mapsto (a_1, \ldots, a_N)$$

is an isomorphism between the additive group L^+ and the direct sum $\oplus_{n=1}^N K^+$, and this is a homeomorphism if the right hand side is given the product topology. In particular, the Haar measures on L^+ and on $\oplus_{n=1}^N K^+$ are the same up to a multiplicative constant in $\mathbf{Q}^*$.

Let $b \in K$. Then the left-multiplication-by-b map

$$b : \sum a_n w_n \mapsto \sum b a_n w_n$$

on L^+ is the same as the map

$$(a_1, \ldots, a_N) \mapsto (ba_1, \ldots, ba_N)$$

on $\oplus_{n=1}^N K^+$, so it multiplies the Haar measure by $|b|^N$, since $|\cdot|$ on K is assumed normalized (the measure of each factor is multiplied by $|b|$, so the measure on the product is multiplied by $|b|^N$). Since $\|\cdot\|$ is assumed normalized, so multiplication by b rescales by $\|b\|$, we have

$$\|b\| = |b|^N.$$

But $b \in K$, so $\mathrm{Norm}_{L/K}(b) = b^N$. Since $|\cdot|$ is nontrivial and for $a \in K$ we have

$$\|a\| = |a|^N = \left|a^N\right| = \left|\mathrm{Norm}_{L/K}(a)\right|,$$

so we must have $c = 1$ in (11.1), as claimed. □

In the case when K need not be complete with respect to the valuation $|\cdot|$ on K, we have the following theorem.

Theorem 11.2. *Suppose $|\cdot|$ is a (nontrivial as always) normalized valuation of a field K and let L be a finite extension of K. Then for any $a \in L$,*

$$\prod_{1 \le j \le J} \|a\|_j = \left|\mathrm{Norm}_{L/K}(a)\right|$$

where the $\|\cdot\|_j$ are the normalized valuations equivalent to the extensions of $|\cdot|$ to K.

Proof. Let K_v denote the completion of K with respect to $|\cdot|$. Write

$$K_v \otimes_K L = \bigoplus_{1 \leq j \leq J} L_j.$$

Then as mentioned at the end of the notes from last time,

$$\mathrm{Norm}_{L/K}(a) = \prod_{1 \leq j \leq J} \mathrm{Norm}_{L_j/K_v}(a). \tag{11.2}$$

We proved last time that the $\|\cdot\|_j$ are exactly the normalizations of the extensions of $|\cdot|$ to the L_j (i.e., the L_j are in bijection with the extensions of valuations, so there are no other valuations missed). By Lemma 10.1, the normalized valuation $\|\cdot\|_j$ on L_j is $|a| = \left|\mathrm{Norm}_{L_J/K_v}(a)\right|$. The theorem now follows by taking absolute values of both sides of (11.2). $\square$

What next?! The last lecture in Math 129 is May 6, so after today there are four more lectures left. There are exactly 8 sections left in Cassels's article, as listed below. Some of these are long, but I will try hard to keep up our pace of two per day, so that we will finish, even if it means omitting a few proofs. *This is worth it because mastery of this material will make it much easier for you to understand much of modern number theory, and also we'll encounter beautiful and more conceptual proofs of the main results of algebraic number theory.* For example, in Section 16 we'll see that finiteness of the class group follows because the class group naturally has the discrete topology and is the continuous image of a compact group. We'll probably skip section 19, which is somewhat long and technical, and finish with Section 18, which includes a deduction of Dirichlet's unit theorem from compactness of the ideles.

12 Global Fields

13 Restricted Topological Products

14 The Adele Ring

15 Strong Approximation Theorem

16 The Idele Group

17 Ideals and Divisors

18 Units

19 Inclusion and Norm Maps for Adeles, Ideles, and Ideals