Math 129: Algebraic Number Theory
Homework Assignment 4

William Stein

Due: Thursday, March 11, 2004

The problems:

1. Find representative ideals for each element of the class group of $\mathbf{Q}(\sqrt{-23})$. Illustrate how to use the Minkowski bound to prove that your list of representatives is complete.

2. Suppose $\mathcal{O}$ is an order in the ring of integers $\mathcal{O}_K$ of a number field. Is every ideal in $\mathcal{O}$ necessarily generated by two elements?

3. Let K be a number field of degree $n > 1$ with s pairs of complex conjugate embeddings. Prove that
$$\left(\frac{\pi}{4}\right)^s \frac{n^n}{n!} > 1.$$

4. Do the exercise on page 19 of Swinnerton-Dyer, which shows that the quantity $C_{r,s}$ in the finiteness of class group theorem can be taken to be $\left(\frac{4}{\pi}\right)^s \frac{n!}{n^n}$.

5. Let α denote a root of $x^3 - x + 2$ and let $K = \mathbf{Q}(\alpha)$. Show that $\mathcal{O}_K = \mathbf{Z}[\alpha]$ and that K has class number 1 (don't just read this off from teh output of the MAGMA `MaximalOrder` and `ClassNumber` commands). [Hint: consider the square factors of the discriminant of $x^3 - x + 2$ and show that $\frac{1}{2}(a + b\alpha + c\alpha^2)$ is an algebra integer if and only if a, b, and c are all even.]

6. If S is a closed, bounded, convex, symmetric set in $\mathbf{R}^n$ with $\mathrm{Vol}(S) \geq m2^n$, for some positive integer m, show that S contains at least $2m$ nonzero points in $\mathbf{Z}^n$.