Math 252: Modular Abelian Varieties

9/22/03

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Today: Constructing Modular Curves

Outline
- $\mathfrak{h} \cong$ open disc
- Fuchsian groups
- $\mathfrak{h}^*$ and its topology
- $\Gamma \backslash \mathfrak{h}^*$
  - topology
  - Riemann surface
  - geometry
- Moduli interpretation of $\Gamma_1(N)\backslash\mathfrak{h}^*$

Note: MAGMA has extensive facilities for computing with congruence subgroups. Generators, etc. Investigate! (In $\underline{\underline{PSL}}_2(\mathbb{Z})$.)

1. $\mathfrak{h} \cong D$;

Lemma: $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ $\cong$ (complex analytic isomorphism) $D = \{z \in \mathbb{C} : |z| < 1\}$

Proof:

$$\mathfrak{h} \xrightarrow[\rho]{z \mapsto \frac{z-i}{z+i}} D$$

(Note that $\left|\frac{z-i}{z+i}\right| < 1$ for $z \in \mathfrak{h}$, since if $z = x+iy$ then

$$|z-i|^2 = x^2 + \underbrace{(y-1)^2 < x^2 + (y+1)^2}_{\text{since } y>0.} = |z+i|^2$$
)

Inverse: $\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ i & -i \end{bmatrix}$. $\quad w \overset{\rho^{-1}}{\mapsto} \dfrac{w+1}{i(w-1)} = i\,\dfrac{w+1}{-w+1}$

$$\mathrm{Im}(\rho^{-1}w) = \mathrm{Im}\left(i\,\frac{w+1}{-w+1}\right) = \frac{1-|w|^2}{|1-w|^2} > 0$$

get this easily.

just use that $\mathrm{Im}(z) = \frac{1}{2i}(z-\bar{z})$ for any $z \in \mathbb{C}$

2. Fuchsian groups

Defn: A subgroup $G \leq SL_2(\mathbb{R})$ is discrete if it has the discrete topology as a subspace of the topological group $SL_2(\mathbb{R})$

Ex: $SL_2(\mathbb{Z})$ is discrete

Defn: A subgroup $G \subset SL_2(\mathbb{R})$ is a Fuchsian group if it is discrete.

Fact: Fuchsian groups Γ act "properly discontinuous" on h^*, so quotients behave well.

If $x, y \in h^*$ $\exists$ U, V s.t. $x \in U$, $y \in V$

$\#\{\gamma \in \Gamma : \gamma U \cap V \neq \emptyset\} < \infty$,

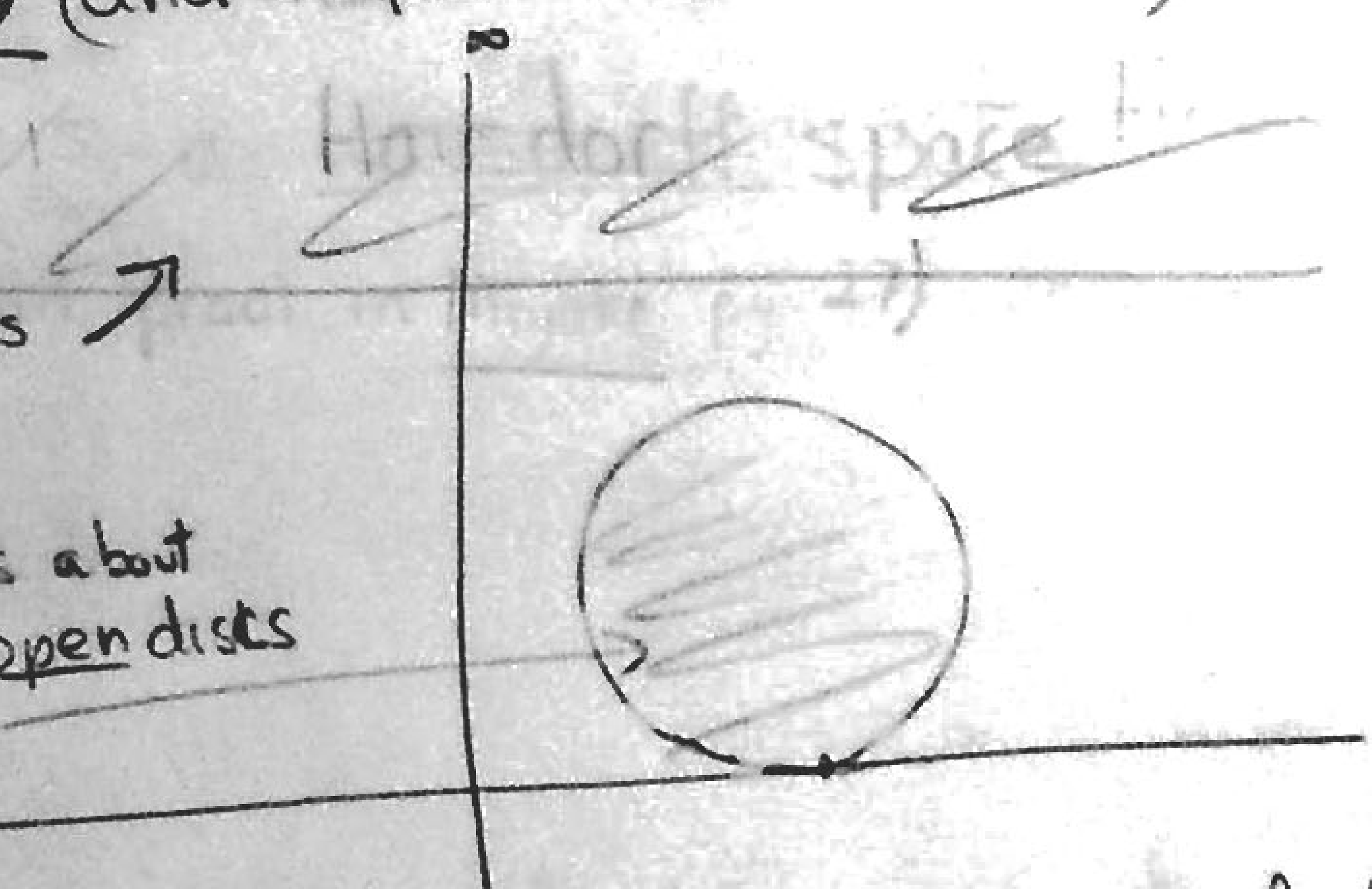
[This is Thm 1.5.2 of Miyake]

3. h^*:

Let $h^* = h \cup \overbrace{\{\mathbb{Q}\} \cup \{\infty\}}^{\mathbb{P}^1(\mathbb{Q})}$.

Give h^* a topology (and complex analytic structure) as follows:

- basis of open sets about ∞ consists of horizontal strips



- basis of open sets about $x \in \mathbb{Q}$ consists of open discs tangent to x

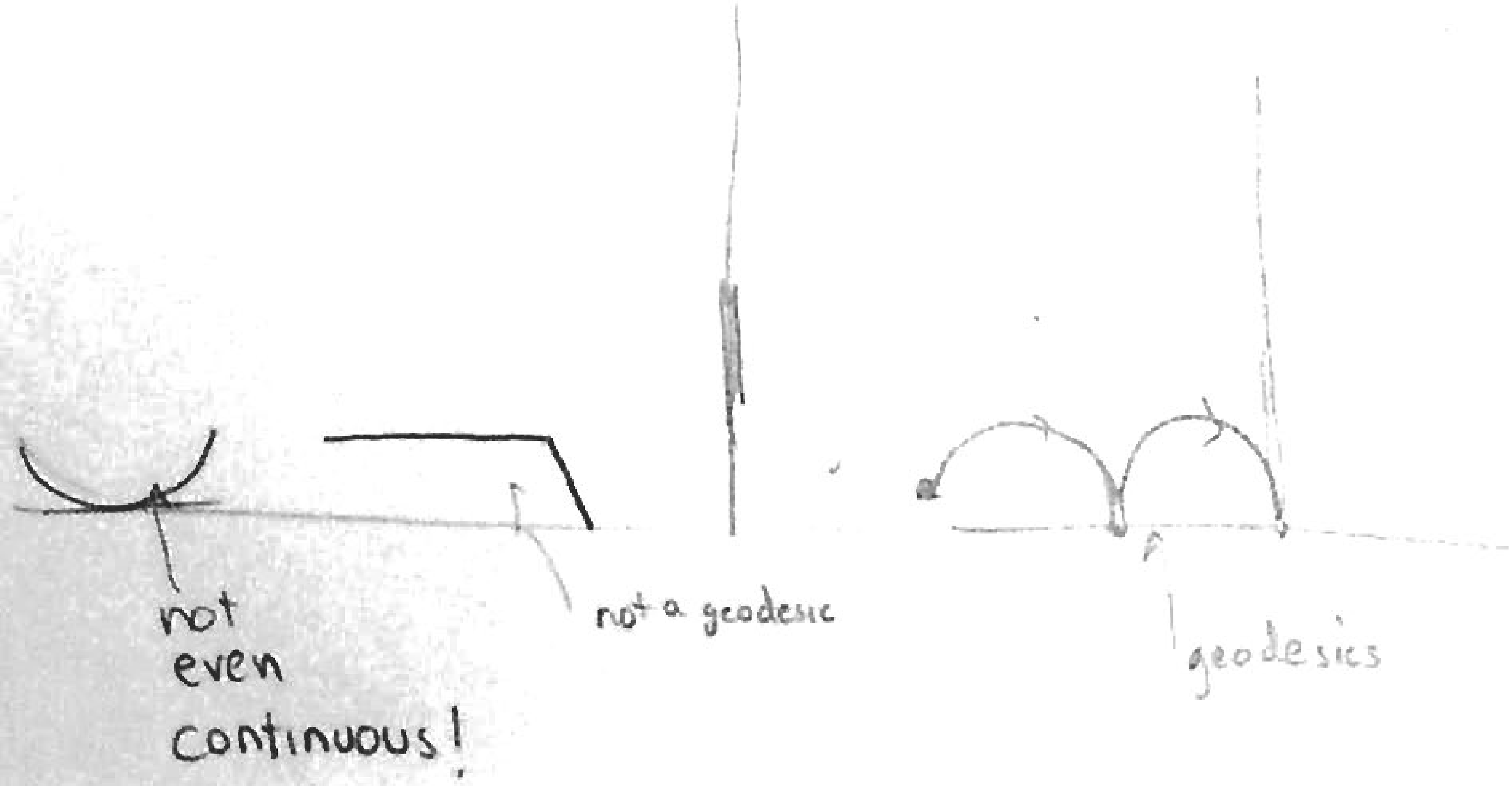
Geometry: Poincaré metric ($\leftrightarrow$ Haar measure on h for $GL_2^+(\mathbb{R})$)

$$ds^2 = \frac{dx^2 + dy^2}{y^2}, \quad \text{so if } \phi : [0,1] \to h,\ C^\infty \text{ map,}$$

$$\text{length}(\phi) = \int_0^1 \frac{\sqrt{(dx(t)/dt)^2 + (dy(t)/dt)^2}}{y(t)}\, dt \in \mathbb{R}.$$

Fact: Geodesics are circles orthogonal to real axis or line orthogonal to real axis.

[Proof: See Miyake, pg. 11–12]



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4. $\Gamma \backslash \mathfrak{h}^*$

Let $\Gamma \subseteq PSL_2(\mathbb{Z})$ be a Fuchsian group.

Facts: $\Gamma \backslash \mathfrak{h}^*$ is a Hausdorff topological space.

(pg 27 of Miyake)

$\Gamma \backslash \mathfrak{h}^*$ is a Riemann Surface, i.e., a 1-dimensional connected complex manifold.

(can be covered by coordinate charts (V_α, t_α) such that transition maps are holomorphic)

Proof is Miyake, pg 28–30 or Shimura pg 17–18. Basically get it from $\mathfrak{h}$.

Three cases:
- ordinary point ($\mathrm{Stab}_\Gamma(z) = \{1\}$). Easy.
- "elliptic point" (finite stabilizer) ← rotation
- cusp $\in \mathbb{P}^1(\mathbb{Q})$
- use $e^{2\pi i z/h}$

5. Modular Curves (as Riemann Surfaces)

Let N be a positive integer.

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : N \mid c \right\}$$

$$X_0(N) = \Gamma_0(N) \backslash \mathfrak{h}^* \longleftarrow \text{always a compact Riemann surface}$$

$$= Y_0(N) \sqcup \underbrace{\Gamma_0(N) \backslash \mathbb{P}^1(\mathbb{Q})}_{\text{cusps}}$$

<u>topologically</u>:

genus = number of holes

<u>geometrically</u>: (?)

<u>Examples</u>:

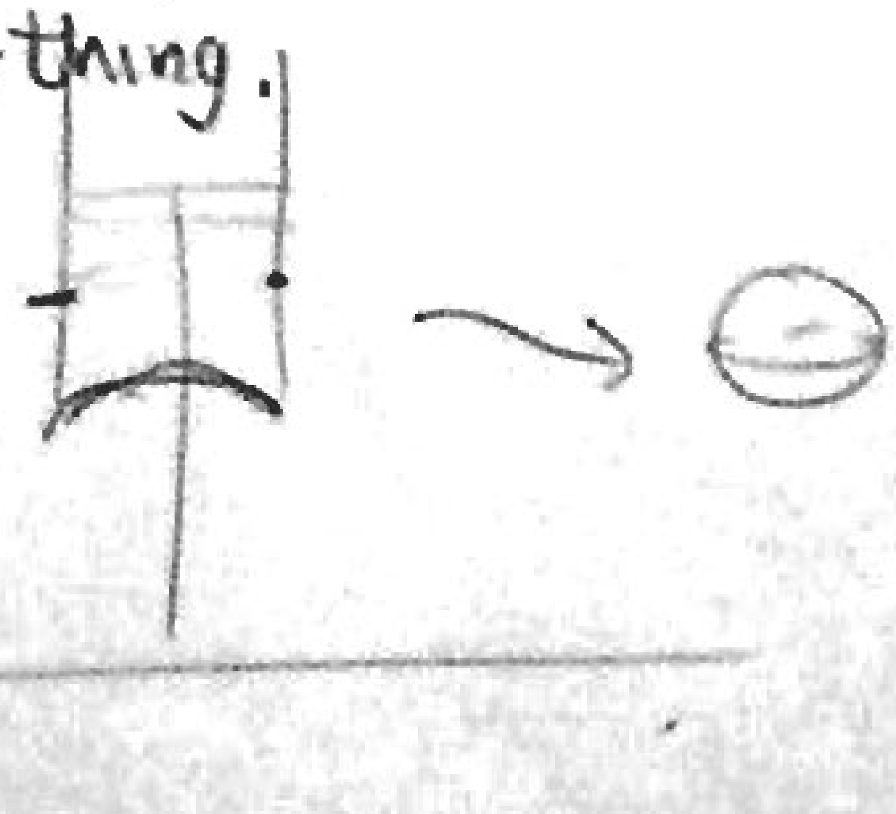
<u>$N=1$</u>:

$X_0(1) \cong$ 

$\#(SL_2(\mathbb{Z}) \backslash \mathbb{P}^1(\mathbb{Q})) = 1.$

genus 0

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} (\infty) = \frac{a}{c}$, so get everything.



We know this because of the fundamental domain in Lecture 2.

On wednesday we'll use this with Riemann-Hurwitz to compute some genus formulas.

N	1 ⋯ 10	11	12	13	14	⋯ 25	100	389	2003
genus of $X_0(N)$	0	1	0	0	1	0	7	32	167

Aside:

grad students w/ me at Berkeley

Theorem (Csirik, Wetherell, Zieve) Lots of results about the function $N \mapsto \text{genus}\, X_0(N)$.

E.g.: • 150, 180, 210, 286, ... are the first few integers that are not the genus of any modular curve.

• genus $X_0(N)$ is odd with probability 1

• $\{\text{genus}\, X_0(N) : \text{all } N\} \subseteq \mathbb{Z}$

density 0 set.

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* $\Gamma_1(N)$ $\quad \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod N$

compact Riemann surface

$X_1(N) := \Gamma_1(N)\backslash\mathfrak{h}^* \supseteq Y_1(N) = \Gamma_1(N)\backslash\mathfrak{h}$.

genus $X_1(N) = 0$ for $N \leq 10$.

genus $X_1(389) = 6112$, genus $X_1(2003) = 166167$ ← formulas next time

* $\Gamma(N)$ $\quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N)$

$X(N) = \Gamma(N)\backslash\mathfrak{h}^*$ ..

we won't consider $X(N)$ too much...

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6. Moduli Interpretation (§7.2 of Diamond-Im)

The points on $Y_0(N)$ and $Y_1(N)$ have a nice moduli-theoretic interpretation.

$$Y_0(N) \equiv \left\{\begin{array}{l}\text{isomorphism classes of pairs} \\ (E, C),\ \text{where } E \text{ is an elliptic} \\ \text{curve an } C \subset E \text{ is a cyclic} \\ \text{subgroup of order } N\end{array}\right\}$$

(<u>Note</u>: Often better to view $Y_0(N)$ as isomorphism classes of cyclic N-isogenies $(E \to F)$ since certain things are more natural.)

<u>Why?</u>

$$\Gamma_0(N)\backslash \mathfrak{h} \ni [\underset{\mathfrak{h}}{\overset{\tau}{\cap}}] \longleftrightarrow \left[\left(\mathbb{C}/_{\mathbb{Z}+\mathbb{Z}\tau},\ \left\langle \tfrac{1}{N}\mathbb{Z}/\mathbb{Z}\right\rangle\right)\right]$$

$$Y_1(N) = \left\{\begin{array}{l}\text{isomorphism classes of pairs} \\ (E, P)\ \text{where } P \text{ is a } \underline{\text{point}} \\ \text{on } E \text{ of exact order } N\end{array}\right\}$$

$$\Gamma_1(N)\backslash \mathfrak{h} \ni [\tau] \longleftrightarrow \left[\left(\mathbb{C}/_{\mathbb{Z}+\mathbb{Z}\tau},\ \tfrac{1}{N}\right)\right]$$

<u>Next time</u>: * Prove that we have these bijections (computational subtlety?)

* Genus formulas.