Math 581g, Fall 2011, Homework 4: SOLUTIONS

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December 12, 2011

1. What is the dimension of M_{40}? Of S_{40}? What is the dimension of S_{2012}?

   Solution.

```
sage: dimension_modular_forms(1,40)
4
sage: dimension_cusp_forms(1,40)
3
sage: dimension_cusp_forms(1,2012)
167
```

2. (a) Write down an explicit basis for S_{40} where each q-expansion is given to precision $O(q^{10})$, i.e., give coefficient of q^9, then $+\cdots$.

   Solution.

```
sage: S = CuspForms(1,40,prec=10)
sage: S.basis()
[
q + 19291168*q^4 + 37956369150*q^5 + 14446985236992*q^6 + 1741415886056000*q^7 + 88205204511072000*q^
q^2 + 156024*q^4 + 57085952*q^5 + 1914094476*q^6 - 27480047616*q^7 - 203825162176*q^8 + 4561693728768
q^3 + 168*q^4 - 12636*q^5 + 392832*q^6 - 7335174*q^7 + 92207808*q^8 - 804651624*q^9 + O(q^10)
]
```

   (b) Consider the q-expansion $f = \sum_{n=1}^{\infty} \mathrm{Tr}(T_n)q^n$, where T_n is the nth Hecke operator on S_{40}. Write f explicitly as a linear combination of your basis from the previous part of this problem.

   Solution.

```
sage: S = CuspForms(1,40,prec=10)
sage: R.<q> = ZZ[['q']]
sage: f = sum([S.T(n).trace()*q^n for n in [1..9]]) + O(q^10); f
3*q + 548856*q^2 + 1109442852*q^3 + 272078981184*q^4 + 17426916500490*q^5 + 1529727847867296*q^6 - 17
sage: g = sum(f[i]*S.basis()[i-1] for i in [1,2,3]); g
3*q + 548856*q^2 + 1109442852*q^3 + 272078981184*q^4 + 17426916500490*q^5 + 1529727847867296*q^6 - 17
```

3. What is the smallest integer B such that the Hecke operators $T_1, T_2, \ldots, T_B$ acting on S_{40} generate the Hecke algebra $\mathbf{T} = \mathbf{Z}[T_1, T_2, \ldots, T_B]$ as a **Z-module**?

   Solution. This should be trivial to do in Sage, but it isn't (I did write a general function that outputs this B once, but only in Magma, evidently.) By pure thought the answer is $B = 3$, because of the special form of the basis. Alternatively, see Prop. 4.5.2 of the course textbook.

4. Make a table that tabulates the characteristic polynomial of the Hecke opeator T_2 acting on the spaces $S_{12}, S_{14}, S_{16}, \ldots, S_{50}$.

   Solution.

```
sage: for k in [12,14,..,50]:
...       print k, CuspForms(1,k).hecke_polynomial(2)
12 x + 24
14 1
16 x - 216
18 x + 528
```

```
20 x - 456
22 x + 288
24 x^2 - 1080*x - 20468736
26 x + 48
28 x^2 + 8280*x - 195250176
30 x^2 - 8640*x - 454569984
32 x^2 - 39960*x - 2235350016
34 x^2 + 121680*x - 8513040384
36 x^3 - 139656*x^2 - 59208339456*x - 1467625047588864
38 x^2 + 194400*x - 137403408384
40 x^3 - 548856*x^2 - 810051757056*x + 213542160549543936
42 x^3 + 344688*x^2 - 6374982426624*x - 520435526440845312
44 x^3 + 2209944*x^2 - 15663522502656*x - 19976984434430705664
46 x^3 - 3814272*x^2 - 44544640241664*x + 135250282417024401408
48 x^4 - 5785560*x^3 - 467142374034432*x^2 + 1426830562183253852160*x + 3297913828840214320807673856
50 x^3 + 24225168*x^2 - 566746931810304*x - 136348832287427364126 72
```

5. Let $\tau(n)$ be the Ramanujan τ function, i.e., the coefficients of the modular form Δ.

   (a) Is $\tau(p) = 0$ for any $p < 10^6$?

   Solution.

   ```
   sage: time f = delta_qexp(10^6)
   Time: CPU 9.93 s, Wall: 10.02 s
   sage: [p for p in primes(10^6) if f[p] == 0]
   []
   ```

   (b) What is the average of the numbers $\tau(p)$ over primes $p < 10^6$? [Hint: You may want to use the function `delta_qexp` to compute all $\tau(n)$ for $n < 10^6$ in a few seconds.]

   Solution.

   ```
   sage: v = [f[p] for p in primes(10^6)]
   sage: float(sum(v))/len(v)
   2.6329152726127548e+29
   sage: sum(v)/len(v)
   492091864451323883085215897841763/1869
   ```

   (c) What about the average of $\tau(p)/p^{11/2}$ over primes $p < 10^6$? (You may compute a floating point approximation.)

   Solution.

   ```
   sage: w = [float(f[p])/float(p)^(11/2) for p in primes(10^6)]
   sage: sum(w)/len(w)
   0.0011803276202013494
   ```