Math 581g, Fall 2011, Homework 3: SOLUTIONS

William Stein (wstein@uw.edu)

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1. (Warm up) Prove that for τ in the upper half plane, we have
$$\overline{(e^{-2\pi i\overline{\tau}})} = e^{2\pi i\tau}.$$

   Solution. Writing $\tau = x + iy$, we have the following straightforward calculation
$$\overline{e^{-2\pi i\overline{\tau}}} = \overline{e^{-2\pi i(x-iy)}} = \overline{e^{-2\pi ix}e^{-2\pi y}} = e^{2\pi ix}e^{-2\pi y} = e^{2\pi i(x+iy)} = e^{2\pi i\tau}.$$

   Notice that the hypothesis that τ is in the upper half plane is not required.

2. Let d and m be positive integers. Prove that $\frac{1}{d}\sum_{b=0}^{d-1}(e^{\frac{2\pi im}{d}})^b$ is only nonzero if $d \mid m$, in which case the sum equals 1.

   Solution. If $d \mid m$, then $e^{\frac{2\pi im}{d}} = 1$, which proves that the sum divided by d is 1. If $d \nmid m$, then $e^{\frac{2\pi im}{d}} \neq 1$, so it is some nontrivial kth root ζ_k of unity of order some divisor $k \mid d$. We have
$$\sum_{b=0}^{d-1}(e^{\frac{2\pi im}{d}})^b = \frac{d}{k}\cdot\sum_{b=0}^{k-1}\zeta_k^b.$$

   Since multiplication by ζ_k permutes the elements of the cyclic group $\langle\zeta_k\rangle$, we see that $\zeta_k \cdot \sum_{b=0}^{k-1}\zeta_k^b = \sum_{b=0}^{k-1}\zeta_k^b$, which since $\zeta_k \neq 1$ implies that $\sum_{b=0}^{k-1}\zeta_k^b = 0$.

3. For $\tau \in \mathbf{C}$, let $E_\tau = \mathbf{C}/(\mathbf{Z}\tau + \mathbf{Z})$. Show that $\#\operatorname{Aut}(E_i) = 4$, $\#\operatorname{Aut}(E_\rho) = 6$, and $\#\operatorname{Aut}(E_{\sqrt{-2}}) = 2$. (Recall that $\rho = e^{2\pi i/3}$.)

   Solution. Observe that $\alpha \in \mathbf{C}$ defines an element of $\operatorname{End}(E_\tau)$ if and only if α sends the lattice $\Lambda_\tau = \mathbf{Z}\tau + \mathbf{Z}$ into itself. First, for $\tau = i$, we have $\alpha(\mathbf{Z}i+\mathbf{Z}) \subset \mathbf{Z}i+\mathbf{Z}$ if and only if $\alpha \in \mathbf{Z}[i]$, so $\operatorname{End}(E_i) = \mathbf{Z}[i]$. Thus $\operatorname{Aut}(E_i) = \operatorname{End}(E_i)^* = \mathbf{Z}[i]^*$, which is the group of order 4 generated by i. The proof for ρ and $\sqrt{-2}$ is exactly the same, noting that $\mathbf{Z}[\rho]$ has unit group of order 6, and all other unit groups of rings of integers of quadratic imaginary fields have order 2.