Exercise Set 1:
G-Modules

Math 582e, Winter 2010, University of Washington

Due Wednesday, January 13, 2010

1. Give a complete detailed proof that the functor $A \mapsto A^G$ on the category of G-modules is left exact.

2. Let $G = S_n$ (the symmetric group) and $A = \mathbf{Z}$ equipped with the trivial G-action, so $s.a = a$ for all $a \in A$. What are $\mathrm{H}^0(G, A)$ and $\mathrm{H}^1(G, A)$?

3. An exact sequence $0 \to A \xrightarrow{f} B \to C \xrightarrow{g} 0$ is *split* if there exists a morphism $h : C \to B$ such that $g \circ h = 1_C$.

   (a) Give an example of a short exact sequence $0 \to A \to B \to C \to 0$ with A, B, and C *finite* abelian groups that is *not* split.

   (b) Prove that if $0 \to A \to B \to C \to 0$ is split, then B is isomorphic to $A \oplus C$.

4. Let $G = \mathrm{GL}_2(\mathbb{F}_2)$ and $A = \mathbb{F}_2 \oplus \mathbb{F}_2$ equipped with the natural left matrix action of G on A. Use cocycles and coboundaries to determine $\mathrm{H}^0(G, A), \mathrm{H}^1(G, A)$, and $\mathrm{H}^2(G, A)$. [[I'm not sure – doing H^2 might be hard?]]

5. Suppose G is a group, B is a G-module, and X is an abelian group. Prove that there is an isomorphism of G-modules:

$$\mathrm{Hom}_{\mathbb{Z}}(B, X) \to \mathrm{Hom}_G(B, \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X))$$

   induced by

$$f \mapsto (b \mapsto (s \mapsto f(s.b))).$$