

Exercise Set 1:

G -Modules

Math 582e, Winter 2010, University of Washington

Due Wednesday, January 13, 2010

1. Give a complete detailed proof that the functor $A \mapsto A^G$ on the category of G -modules is left exact.
2. Let $G = S_n$ (the symmetric group) and $A = \mathbf{Z}$ equipped with the trivial G -action, so $s.a = a$ for all $a \in A$. What are $H^0(G, A)$ and $H^1(G, A)$?
3. An exact sequence $0 \rightarrow A \xrightarrow{f} B \rightarrow C \xrightarrow{g} 0$ is *split* if there exists a morphism $h : C \rightarrow B$ such that $g \circ h = 1_C$.
 - (a) Give an example of a short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ with A , B , and C *finite* abelian groups that is *not* split.
 - (b) Prove that if $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is split, then B is isomorphic to $A \oplus C$.
4. Let $G = \text{GL}_2(\mathbb{F}_2)$ and $A = \mathbb{F}_2 \oplus \mathbb{F}_2$ equipped with the natural left matrix action of G on A . Use cocycles and coboundaries to determine $H^0(G, A)$, $H^1(G, A)$, and $H^2(G, A)$. [[I'm not sure – doing H^2 might be hard?]]
5. Suppose G is a group, B is a G -module, and X is an abelian group. Prove that there is an isomorphism of G -modules:

$$\text{Hom}_{\mathbb{Z}}(B, X) \rightarrow \text{Hom}_G(B, \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], X))$$

induced by

$$f \mapsto (b \mapsto (s \mapsto f(s.b))).$$