Noncommutative Dedekind Domains

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In this paper we will discuss noncommutative analogs of Dedekind Domains. First we give the definition of a Dedekind Domain for the commutative case.

Definition. A *(commutative) Dedekind Domain* is a Noetherian, integrally closed, integral domain of Krull dimension less than or equal to 1.

When R is a (commutative) Dedekind Domain and K is its field of fractions then we get the following nice properties:

1. Every nonzero fractional ideal of R in K is invertible.
2. Every nonzero proper ideal I of R can be written as a finite product of (not necessarily distinct) prime ideals. And this factorization is unique up to reordering.
3. Every nonzero fractional ideal of R in K is a projective R-module.
4. Arithmetic of ideals.
5. Every ideal of R can be generated by two elements.

When we look for noncommutative analogs to Dedekind Domains, we will be looking for noncommutative rings with similar properties. When we work in the noncommutative world, the word ideal means two-sided ideal and the word Noetherian means both left and right Noetherian.

Before we dive into definitions, let's take a look at a family of rings that we will use as an example throughout this paper.

Example. Weyl Algebras

Let k be a field of characteristic 0. Let $A_n(k)$ be the k-algebra with generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and relations $x_i y_j - y_j x_i = \delta_{ij}$ and $x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0$. $A_n(k)$ is called the nth Weyl Algebra.

Applying the relations repeatedly we get the forumlas

$$x_i y_i^m - y_i^m x_i = m y_i^{m-1} = \frac{\partial}{\partial y_i} y_i^m$$

$$x_i^n y_i - y_i x_i^n = n x_i^{n-1} = \frac{\partial}{\partial x_i} x_i^n$$

which will be useful later on.

There are different ways to think about $A_n(k)$. One way is to look directly at the relations. For example, $A_1(k)$ is like the polynomial ring $k[x, y]$ except that x and y don't commute. Instead we get the relation $yx = xy - 1$. So $A_1(k)$ is not commutative, but it is noncommutative in a nice way. We see that if we have a monomial $x^{n_1} y^{n_2} x^{m_1} y^{m_2}$ in $A_1(k)$ we can use the relation to rewrite this as $x^{n_1+m_1} y^{n_2+m_2}$+(lower order terms).

Another way to view $A_n(k)$ is as a skew polynomial ring. So we pause here to make a quick digression into the world of skew polynomial rings. Let R be a ring and let δ be a derivation on R (that is, a map $\delta : R \to R$ that satisfying $\delta(ab) = \delta(a)b + a\delta(b)$ and $\delta(a+b) = \delta(a)+\delta(b)$ for all $a, b \in R$. Then the skew polynomial ring $S = R[y; \delta]$ is a ring that contains R as a subring, and contains an element y with relation $yr = ry + \delta(r)$ for all $r \in R$. Now we state a couple of useful lemmas about skew polynomial rings.

Lemma 1. If R is an integral domain, then $S = R[y; \delta]$ is also an integral domain.

Proof. See [4].

Lemma 2. If R is right (left) Noetherian, then $S = R[y; \delta]$ is also right (left) Noetherian.

Proof. See [4].

We can view $A_n(k)$ as skew polynomial ring R_n, defined as follows. Let $R = k[x_1, \dots, x_n]$ be the usual polynomial ring. We define R_n inductively: $R_0 = R$, $R_{i+1} = R_i[y_{i+1}; \frac{\partial}{\partial x_{i+1}}]$. By basic calculus we know that $\frac{\partial}{\partial x_{i+1}}$ satisfies the definition of a derivation. Direct computation shows that $A_n(k) \cong R_n$.

Using the above tells us that each element of $A_n(k)$ can be written uniquely in the form $\sum a_{\alpha\beta} x^\alpha y^\beta$ where $\alpha = (m_1, \dots, m_n)$, $\beta = (p_1, \dots, p_n)$, $x^\alpha = x_1^{m_1} \cdots x_n^{m_n}$, and $y^\beta = y_1^{p_1} \cdots y_n^{p_n}$. This standard form is often useful for computations. For example, because we can write every element of $A_1(k)$ uniquely as $\sum a_{ij} x^i y^j$ then we can define the degree in x of an element to be the highest power of x that appears when the element is written in this standard form. And we can define the degree in y of an element similarly.

Now we will proceed with defining noncommutative Dedekind domains. Our first step will be to modify the definition of a prime ideal. In the commutative case, we have the following definition:

Definition. If R is a commutative ring, an ideal P is a *prime ideal* if P is a proper ideal of R and whenever $ab \in P$, for $a, b \in R$, then either $a \in P$ or $b \in P$.

If we define prime ideals of noncommutative rings using this definition then there are rings that will not have any prime ideals.

Example. Let $R = M_n(k)$, where k is a field and $n \geq 2$. Let $E_{ij} \in R$ be the element with a 1 in the (i, j) position and 0's elsewhere. Then $E_{11}E_{22} = 0$ but neither are in the ideal 0. So if we use the commutative definition of prime ideal, then 0 is not a prime ideal. But R is a simple ring (that is, the only proper (two-sided) ideal of R is 0) so with this definition R has no prime ideals.

So, for the noncommutative case this definition of prime ideal does not turn out to be very useful. Instead we use the following definition of a prime ideal:

Definition. If R is a ring, an ideal P is a *prime ideal* if P is a proper (two-sided) ideal of R and whenever $IJ \subset P$, for ideals I, J in R, then either $I \subset P$ or $J \subset P$.

If R is a commutative ring, this definition is equivalent to the previous definition. But for noncommutative rings, this definition turns out to be much more useful than the previous definition. Note here that we use the common strategy for generalizing commutative concepts to the noncommutative setting of replacing elements with ideals.

For noncommutative rings, we keep the same definition of integral domain. That is,

Definition. A ring R (either commutative or noncommutative) is called an *integral domain* if the product of two nonzero elements is always nonzero.

But in the noncommutative setting we replace the *concept* of an integral domain with concept of a prime ring, defined as follows:

Definition. A ring is a *prime ring* if 0 is a prime ideal. Equivalently, if I and J are nonzero ideals of R, then IJ is a nonzero ideal of R.

Here again we have used the general strategy of replacing elements by ideal to generalize from the commutative case to the noncommutative case.

If R is a commutative ring, then R is an integral domain if and only if R is a prime ring. For the noncommutative case, every integral domain is a prime ring. But a noncommutative prime ring is not necessarily an integral domain.

Example. Let $R = M_n(k)$ with $n \geq 1$. Let I and J be non zero ideals of R. Let $A \in I$ and $B \in J$ be nonzero elements, say $(A)_{ij}$ and $(B)_{kl}$ are nonzero entries in A and B. Then $AE_{jk}B$ has a nonzero entry in the (i, l) position. And $AE_{jk}B \in IJ$. So we see that IJ is a nonzero ideal. So R is a prime ring. But we already saw that R is not an integral domain.

Proposition. $A_n(k)$ is a prime ring.

Proof. We prove that $A_n(k)$ is a prime ring by proving that is it an integral domain. We use the skew polynomial definition of $A_n(k)$ and induction on n. For $n = 0$ we have that R_0 is a commutative polynomial ring, so it is an integral domain. Now assume that R_i is an integral domain. Then $R_{i+1} = R_i[y_{i+1}; \frac{\partial}{\partial x_{i+1}}]$ is an integral domain, by lemma 1. So R_n is an integral domain, so $A_n(k) \cong R_n$ is a prime ring. ∎

Note for the particular case of $A_1(k)$ we can see that it is an integral domain in a more direct manner. Let A and B be nonzero elements in $A_1(k)$. Then we can write $A = \sum_{i=0}^{n} f_i y^i$ and $B = \sum_{j=0}^{m} g_j y^j$ where f_i, $g_j \in k[x]$ and f_n, g_m are nonzero. Then

$$f_n y^n g_m y^m = f_n g_m y^{n+m} + \text{(lower order terms)}$$

and $f_n g_m$ is nonzero because $k[x]$ is an integral domain. All other terms in AB will have degree in y of less than $n + m$, so no other term can cancel out this nonzero term. So $AB \neq 0$.

Definition. A ring R is *right Goldie* if

1. R does not contain an infinite direct sum of non-zero right ideals, and

2. R satisfies the ascending chain condition on right annihilators (that is, if we have an increasing chain of right annihilators $A_1 \subset A_2 \subset \cdots$ then this chain eventually stablizes).

A left Goldie ring is defined similarly. A ring is called *Goldie* if it is both left and right Goldie.

We won't concern ourselves with the precise definition of Goldie rings. We will just note that Noetherian rings are Goldie rings.

Proposition. A right (left) Noetherian ring is a right (left) Goldie ring.

Proof. Let R be a right (left) Noetherian ring. Then condition (1) is satisfied because if such an infinite direct sum of non-zero right (left) ideals existed, then the ring would have an ascending chain of right (left) ideals that does not stablize. And condition (2) is satisfied because right (left) annihilators are right (left) ideals. ■

Proposition. $A_n(k)$ is a Goldie ring.

Proof. We prove that $A_n(k)$ is a Goldie ring by proving that it is Noetherian. Again, we use the skew polynomial definition of $A_n(k)$ and induction on n. For $n = 0$ we have that R_0 is a commutative polynomial ring, so it is left and right Noetherian. Now assume that R_i is left and right Noetherian. Then $R_{i+1} = R_i[y_{i+1}; \frac{\partial}{\partial x_{i+1}}]$ is Noetherian, by lemma 2. So R_n is Noetherian, so $A_n(k) \cong R_n$ is a Goldie ring. ■

Definition. An ideal I of a ring R is *invertible* if $AB = BA = R$ for some subset B of some extension ring S of R.

Definition. A ring R is an *Asano prime ring* if it is a prime Goldie ring such that every non-zero ideal of R is invertible.

The following theorem shows us that Asano prime rings have nice properties of their (two-sided) ideals.

Theorem. Every nonzero ideal of an Asano prime ring R is the unique commutative product of maximal ideals.

Proof. First we show that the multiplication of maximal ideals is commutative. Let M_1 and M_2 be distinct nonzero maximal ideals in R. Let $X = M_1^{-1}(M_1 \cap M_2)$ (such an M_1^{-1} exists because R is an Asano ring). Then $X = M_1^{-1}(M_1 \cap M_2) \subset M_1^{-1}(M_1) = R$. So X is an ideal in R. And $M_1 X = M_1 M_1^{-1}(M_1 \cap M_2) = M_1 \cap M_2 \subset M_2$. So we get that $M_1 X \subset M_2$. But M_2 is a maximal ideal, so it is also prime. So either $M_1 \subset M_2$ or $X \subset M_2$. But the first cannot happen because M_1 and M_2 are distinct maximal ideals. So $X \subset M_2$. But this means that $M_1 \cap M_2 = M_1 X \subset M_1 M_2 \subset M_1 \cap M_2$. So we must have equality throughout. So $M_1 M_2 = M_1 \cap M_2$. But the same argument shows $M_2 M_1 = M_2 \cap M_1$. So we get $M_1 M_2 = M_2 M_1$. According to [4] the "proof is now routine". So we take the authors at their word. ■

Proposition. $A_n(k)$ is simple.

Proof. Let I be a nonzero two-sided ideal in $A_n(k)$. Let c be a nonzero element in I. Then $x_i c - c x_i$ and $c y_i - y_i c$ are elements in I. Recall that x_i commutes with x_j for any j and with y_j for $i \neq j$. Then using the formulas computed previously, we get

$$x_i c - c x_i = \frac{\partial c}{\partial y_i}$$

Similarly we get

$$c y_i - y_i c = \frac{\partial c}{\partial x_i}$$

So these partial derivatives of c are in I. Now we can apply these partial derivatives to c until all we get a nonzero element a of k. Then $a \in I$, so $I = A_n(k)$. So $A_n(k)$ is simple. ■

We have already seen that $A_n(k)$ is a prime Goldie ring. This proposition shows us that $A_n(k)$ itself is the only nonzero ideal of $A_n(k)$, so every nonzero ideal of $A_n(k)$ is invertible. So $A_n(k)$ is an Asano prime ring.

$A_n(k)$ may not have any (two-sided) ideals other than 0 and itself, but this does not mean that the set of one-sided ideals has such a simple structure.

Proposition. Let $R = A_1(k)$ and let I be the right ideal generated by $1 + xy$ and x^2. Then I is not a principal ideal.

Proof. Write $I = (1 + xy)R + x^2 R$. Suppose $I = aR$ for some $a \in I$. Then there exists some $b \in R$ such that $ab = 1 + xy$ and some $c \in R$ such that $ac = x^2$.

Then, as we saw in the discussion that showed $A_1(k)$ was an integral domain, the degrees in x and y behave well with respect to multiplication. So we know that the degree in y of a is 0 and degree in x of a must be less than or equal to 1.

But by direct computation we see that $(1 + xy)x = x^2 y$. So we can actually write $I = (1 + xy)k[y] + x^2 R$. We can easily use this to write every element of I in standard form and we see that every element of I has degree in x ~~of~~ greater than 1. So, in particular, we get that degree in x of a is 1. So $a = a_0 + a_1 x$, where $a_i \in k$ and $a_1 \neq 0$. Then using similar degree arguments we get that degree in x of b is 0 and degree in y of b is 1. So $b = b_0 + b_1 y$, where $b_i \in k$ and $b_1 \neq 0$. But then

$$1 + xy = ab = (a_0 + a_1 x)(b_0 + b_1 y) = a_0 b_0 + a_0 b_1 y + a_1 b_0 x + a_1 b_1 xy$$

which is not possible when $a_1, b_1 \neq 0$. So no such a exists. So I is not a principal right ideal. ■

Nice properties of (two-sided) ideals will not be very useful for simple rings. And in fact, we will see that many of the rings we will be interested in are simple. So now we add more conditions on R so that we get rings whose one-sided ideals have Dedekind-like properties.

Definition. A ring R is called *(right) hereditary* if every right ideal is projective.

Exercise. A commutative integral domain R is hereditary if and only if R is a Dedekind Domain.

Definition. Let R be a ring, let I be a right ideal of R, and let $pd(R/I)$ be the projective dimension of R/I. The *right global dimension* of a ring R, written $r.gl.dim(R)$, is $sup\{pd(R/I) : I \text{ is a right ideal of } R\}$.

Proposition. $r.gl.dim(A_n(k)) = n$

Proof. See [4].

Theorem. A ring R is right hereditary if and only if $r.gl.dim(R) \leq 1$.

Proof. First assume that R is right hereditary. Let I be a right ideal of R. Then we have the short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

But R is always projective and by assumption I is projective, so this is a projective resolution. So $pd(R/I) \leq 1$, so $r.gl.dim(R) \leq 1$.

Now assume that $r.gl.dim(R) \leq 1$. Let I be a right ideal of R. Then by assumption we have a short exact sequence

$$0 \to P_1 \to P_0 \to R/I \to 0$$

where the P_i's are projective. And we also have the short exact sequence

$$0 \to I \to R \to R/I \to 0.$$

So by Schanuel's Lemma (because R and P_0 are projective) we have that $P_0 \oplus I \cong R \oplus P_1$. Then because $R \oplus P_1$ and P_0 are projective we get that I is projective. So R is hereditary. ■

So in some sense we are replacing the Krull dimension condition for commutative Dedekind domains with a condition on right global dimension.

Proposition. $A_1(k)$ is hereditary.

Proof. This is true by the previous proposition and theorem. ■

Definition. A ring R is a *Dedekind prime ring* if it is a hereditary Noetherian Asano prime ring.

Dedekind prime rings have one-sided ideals that behave as in Dedekind Domains. Dedekind prime rings also have the nice Dedekind-like property that every right ideal can be generated by 2 elements.

Example. By our previous work, we see that $A_1(k)$ is a Dedekind prime ring.

Example. Classical maximal orders over commutative Dedekind domains are Dedekind prime rings. See [4] for details.

Definition. A ring R is a *noncommutative Dedekind domain* if it is a Dedekind prime ring and an integral domain.

(Note: I don't know how standard this definition of noncommutative Dedekind domains is. But almost every paper or textbook I found that mentioned this idea referenced [4], which is where I got the definition.)

If we unwind all of these definitions we see that we can more directly define a noncommutative Dedekind domain to be a Noetherian integral domain with $r.gl.dim(R) \leq 1$, such that ~~very~~ nonzero ideal is invertible. And we can also more directly define a Dedekind prime ring to be a Noetherian prime ring with $r.gl.dim(R) \leq 1$, such that ~~very~~ nonzero ideal is invertible.

So we see that $A_1(k)$ is a noncommutative Dedekind domain. It is interesting to note that $A_1(k)$, which seems to be closely related to $k[x, y]$, is a noncommutative Dedekind domain, while $k[x, y]$ itself is not a Dedekind domain. In some ways this should not be surprising because we have seen how different the (two-sided) ideal structures of these two rings are, $A_1(k)$ being a simple ring and $k[x, y]$ having classical Krull dimension 2. And we note that $k[x, y]$ is an integrally closed, Noetherian integral domain, so its classical Krull dimension is the only way it fails to be a Dedekind domain.

We conclude this paper by discussing an interesting result from [2]. The authors comment that when Dedekind domains were extended to the noncommutative world all of the examples were either classical orders or simple rings. They then go on to determine under what conditions this dichotomy holds. They prove the following theorem:

Theorem. If R is a (commutative or noncommutative) Dedekind domain that is a finitely generated algebra over an uncountable, algebraically closed field, then R is either simple or commutative.

The authors then go on to examine this dichotomy for various (weaker) conditions on the ring or different conditions on the field.

References

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