The Riemann Hypothesis

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Introduction

Prime numbers have fascinated mathematicians since the time of the ancient Greeks and are the subject of many questions they asked about numbers. One thing they asked is, "Are there infinitely many primes?" Euclid (300 BC) gave one of the most elegant proofs in mathematics to answer this question.

Suppose there are only a finite number of primes. Let $p_1, p_2, \ldots p_n$ be the list of n distinct primes where $p_1 < p_2 < \ldots < p_n$. We will multiply all of the primes together, add one and call the result N.

$$N = p_1 p_2 p_3 \cdots p_n + 1$$

The number N is either a composite number or a prime number. If N is prime, then we have found a new prime that is not in our list because N is clearly larger than p_n (and therefore larger than p_i for any i). If N is composite, then it must have some prime factor q that divides it. Any $q = p_i$ we choose leaves a remainder of 1 when we try to divide N by it. So, there must be some new prime q, not in our list of primes, that divides N. In either case, we have found a new prime that is not in our list and have reached a contradiction. We must conclude that there are infinitely many prime numbers.

Once it was known that there are infinitely many prime numbers, other questions arose about the primes. A natural next step was to ask, "How many primes are less than 100, 1,000 or 1,000,000?" The German mathematician Carl Friedrich Gauss considered this question about the distribution of prime numbers when he was fourteen years old [Devlin, 2003]. Gauss once told a friend that, when he had spare time, he counted the number of primes in a range of 1,000 numbers. He had counted all of the primes up to 3 million by the end of his life [Sabbagh, 2004].

Through his investigation into the distribution of prime numbers, Gauss noticed a pattern. He observed that the number of primes grow much more slowly than the integers and that this growth is very similar to the way that logarithms behave. A table of logarithms that Gauss had obtained at fourteen was found in his papers. On the back of this table, Gauss had written [Sabbagh, 2004]

$$\text{Primzahlen unter } a \ (= \infty) \ a/\ln a.$$

This statement was Gauss' guess for how the prime numbers are distributed among the integers. It says that the number of primes less than a is approximated by $a/\ln(a)$ and that the relative error of this approximation approaches 0 as a approaches infinity [Wikipedia, 2010a]. Eventually, this became known as the Prime Number Theorem, whose modern notation is

$$\pi(x) \sim \frac{x}{\ln x}.$$

In this notation, $\pi(x)$ represents the actual number of primes less than or equal to x. Coincidentally, the French mathematician Adrien-Marie Legendre also made this conjecture around the same time [Wikipedia, 2010a].

How closely does $x/\ln x$ estimate the number of primes less than a given number? The function $x/\ln x$ estimates that there are 434,294,481 prime numbers less than 10,000,000,000, but the correct number is 455,052,511 primes. This estimate is off by 20,758,030 primes, which may seem like a lot, but it is only 4.56% [Sabbagh, 2004]. The Prime Number Theorem says that this percent error will decrease as x increases, as is evident in Table 1.

x	$\pi(x)$	$x/\ln(x)$	difference	relative error
10^2	25	21	4	16.00%
10^3	168	144	24	14.29%
10^4	1229	1085	144	11.72%
10^5	9592	8685	907	9.46%
10^6	78498	72382	6116	7.79%
10^7	664579	620420	44159	6.64%
10^8	5761455	5428681	332774	5.78%
10^9	50847534	48254942	2592592	5.10%
10^{10}	455052511	434294481	20758030	4.56%
10^{11}	4118054813	3948131653	169923160	4.13%
10^{12}	37607912018	36191206825	1416705193	3.77%

Table 1: Relative error between $x/\ln x$ and $\pi(x)$ [1]

Later, Gauss provided a better estimate to $\pi(x)$, using the logarithmic integral.

$$Li(x) = \int_2^x \frac{dt}{\ln t}$$

As Table 2 shows, $Li(x)$ overestimates the number of primes less than 10^{10} by only 3,101, which is a good improvement over $x/\ln x$. Can an even better estimate be found for the number of primes less than a given number? In the 19th century, a possible answer to this question was published in a short eight-page paper by the German mathematician Georg Friedrich Bernhard Riemann.

[1] The values in Table 1 were generated using Sage.

x	$\pi(x)$	$Li(x)$	difference	relative error
10^2	25	29	4	16.0000%
10^3	168	176	8	4.7619%
10^4	1229	1245	16	1.3019%
10^5	9592	9628	36	0.3753%
10^6	78498	78626	128	0.1631%
10^7	664579	664917	338	0.0509%
10^8	5761455	5762208	753	0.0131%
10^9	50847534	50849233	1699	0.0033%
10^{10}	455052511	455055612	3101	0.0007%
10^{11}	4118054813	4118066388	11575	0.0003%
10^{12}	37607912018	37607950205	38187	0.00001%

Table 2: Relative error between $Li(x)$ and $\pi(x)$ [2]

A Surprising Conjecture

In 1859, Riemann wrote his famous paper, *Ueber die Anzahl der Primzahlen unter einer gegebenen Grösse (On the Number of Primes less than a Given Magnitude)*, which contains one of the most stunning conjectures in all of mathematics. Riemann was studying a particular function of a complex variable.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

Riemann proved that this function, where s is a complex number ($s = x+yi$ and $x, y \in \mathbb{R}$), can be continued analytically to an analytic function over the whole complex plane (with the exception of $s = 1$). This function, $\zeta(s)$, is known as the Riemann zeta function. In his paper, Riemann was looking at the solutions to $\zeta(s) = 0$ and observed that there are zeros at every negative even integer. These are called the trivial zeros of the zeta function. Any zeros other than these are called the non-trivial zeros. There are infinitely many non-trivial zeros and Riemann made the astonishing conjecture that all non-trivial zeros have the form $\frac{1}{2} + yi$ for some real number y. In other words, all the non-trivial zeros lie on the line $x = \frac{1}{2}$, known as the critical line (see Figure 1).

[2]The values in Table 2 were generated using Sage.

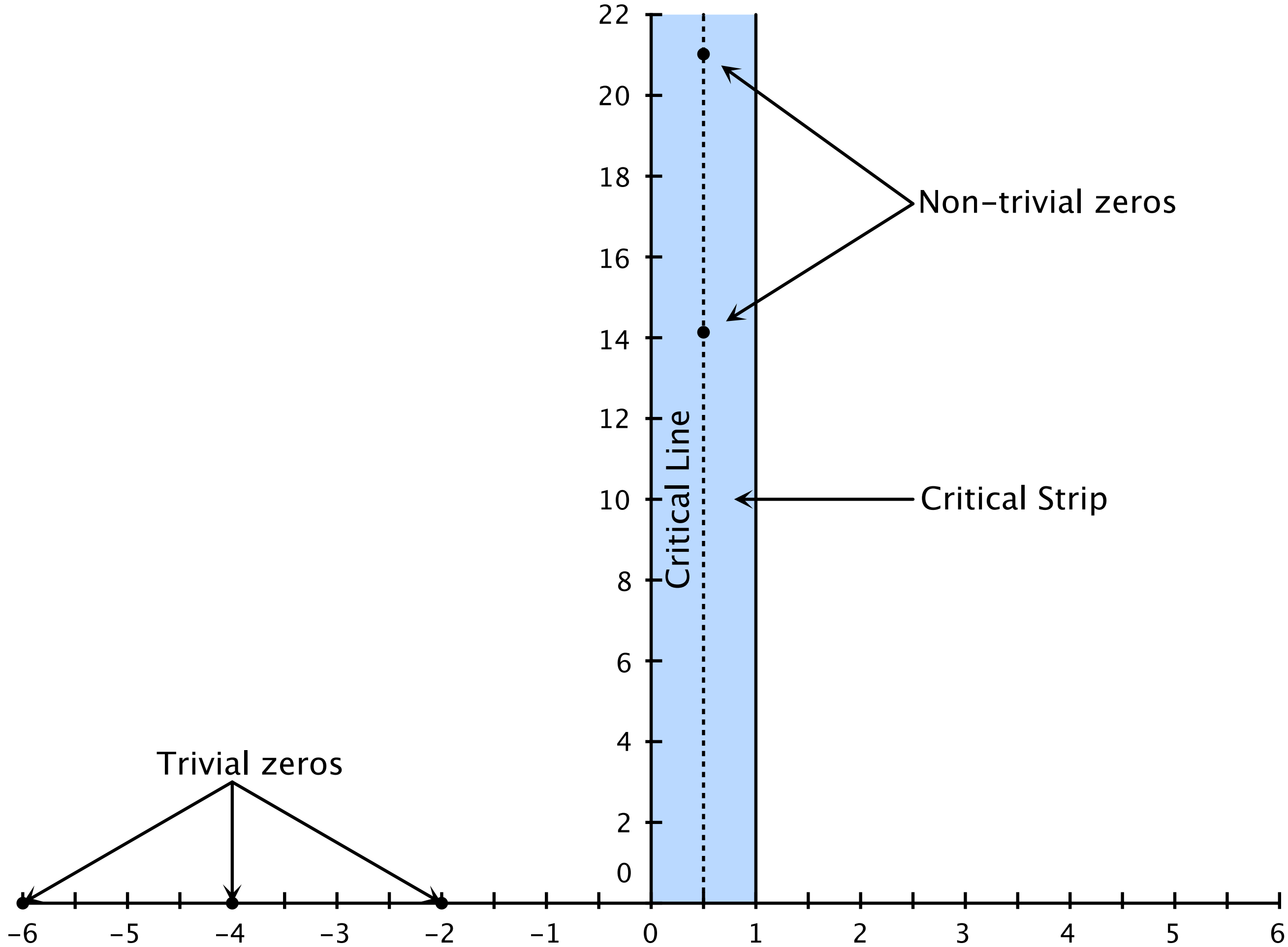

Figure 1: Graph showing the critical strip and the critical line

Why is this important and what does it have to do with the distribution of prime numbers? Riemann made a connection between the non-trivial zeros of his zeta function and the distribution of the prime numbers. Assuming that his conjecture was correct, Riemann provided an even better estimate in his paper for the number of primes less than a given number. In fact, he provided a formula that can compute the exact number of primes without any error. Riemann was not able to provide a proof of his conjecture, saying

> 'Of course, it would be desirable to have a rigorous proof of this; in the meantime, after a few perfunctory vain attempts, I temporarily put aside looking for one, for it seemed unnecessary for the next objective of my investigation.' [Devlin, 2003]

His conjecture states that all non-trivial zeros of $\zeta(s)$ lie on the critical line $\Re(s) = \frac{1}{2}$ and is now known as the Riemann hypothesis. Riemann's conjecture has been considered one of the most important unsolved problems in all of mathematics for at least a century.

Flawed Proofs

There have been several failed attempts at proving the Riemann hypothesis, some of which were publicly announced. In 1885, Thomas Joannes Stieltjes claimed to have proved Mertens' conjecture, which implies the Riemann hypothesis. He died without publishing his proof. There must be a flaw in his argument because, in 1985, Andrew Odlyzko and Herman J.J. te Riele proved that Mertens' conjecture is false [Borwein et al., 2008].

The strangest of these flawed proofs was presented in 1959, when the famous mathematician John Nash gave a lecture at Columbia University sponsored by the American Mathematical Society. Nash was presenting his proof of the Riemann hypothesis to 250 attendees who had high expectations, but unfortunately, his lecture was complete nonsense. This was later explained by his battle with schizophrenia [Sabbagh, 2004].

There have been claims of proofs that show the Riemann hypothesis is false as well. Sabbagh discusses such a claim written about in a *Time* magazine article in 1943. Earlier that year, the editor of the American Mathematical Society's *Transactions* received a wire from the society's secretary asking him to hold the presses for a paper proving that the Riemann hypothesis is false. The author, Hans Rademacher, sent a letter soon after reporting that his calculations had been checked and confirmed by Carl Siegel of Princeton's Institute for Advanced Study. However, at the last moment, Rademacher sent a wire to the editor indicating that Siegel had found a mistake in his reasoning. Rademacher mistakenly relied on the logarithm of a complex number having a unique value, but complex logarithms produce infinitely many values. His proof could not be repaired [Sabbagh, 2004].

Zeroing in on the Zeros

Numerous mathematicians have attempted to prove the Riemann hypothesis over the last 150 years, but most have made little progress. It is reasonable to ask, "Do we have any reason to believe the Riemann hypothesis is true?" There is an enormous amount of empirical evidence to suggest that Riemann's conjecture is correct. From the time he first formulated his hypothesis, mathematicians have been computing zeros of the $\zeta(s)$ and have found that they all lie on the critical line. Riemann himself performed computations of the first few zeros prior to the presentation of his paper. Riemann's calculations were never published and his method was not known until the German number theorist Carl Siegel discovered them while studying Riemann's notes. In the 1930s, Siegel published the formula that Riemann used for calculating zeros of $\zeta(s)$ and it became known as the Riemann-Siegel formula. So far, all large-scale computations of $\zeta(s)$ have been based on it. As of 2004, the first ten trillion non-trivial zeros have been computed and they all lie on the critical line, but these calculations do not constitute a proof [Borwein et al., 2008]. The table below shows the history of these computations, which provide evidence supporting the Riemann hypothesis.

Year	Number of zeros	Computed by
1859 (approx.)	1	B. Riemann
1903	15	J. P. Gram
1914	79	R. J. Backlund
1925	138	J. I. Hutchinson
1935	1,041	E. C. Titchmarsh
1953	1,104	A. M. Turing
1956	15,000	D. H. Lehmer
1956	25,000	D. H. Lehmer
1958	35,337	N. A. Meller
1966	250,000	R. S. Lehman
1968	3,500,000	J. B. Rosser, et al.
1977	40,000,000	R. P. Brent
1979	81,000,001	R. P. Brent
1982	200,000,001	R. P. Brent, et al.
1983	300,000,001	J. van de Lune, H. J. J. te Riele
1986	1,500,000,001	J. van de Lune, et al.
2001	10,000,000,000	J. van de Lune (unpublished)
2004	900,000,000,000	S. Wedeniwski
2004	10,000,000,000,000	X. Gourdon

Table 3: Computations of the zeros of the Riemann zeta function [3]

Some mathematicians have provided small steps forward on the theoretical side as well. In 1896, the French mathematician Jacques Salomon Hadamard and the Belgian mathematician Charles Jean de la Vallée-Poussin independently proved that no zeros could lie on the line x = 1. They also showed that all non-trivial zeros of the Riemann zeta function must lie in the interior of the strip $0 < x < 1$, known as the critical strip (see Figure 1). This was a key step in their proof of the Prime Number Theorem [Sabbagh, 2004].

In the 20th century, mathematicians continued to narrow down the location of the non-trivial zeros. In 1914, G.H. Hardy proved a necessary condition for the Riemann hypothesis being true. He proved that there are infinitely many zeros on the critical line. Mathematicians continued to approach the Riemann hypothesis in this way, by proving stronger results on how many zeros lie on the critical line. In 1942, Atle Selberg proved that at least a small, positive proportion of zeros lie on the critical line (a result that helped him to earn the Fields Medal in 1950). In 1974, Norman Levinson improved this result and showed that one-third of the zeros lie on the critical line. The best and most recent result was proved by Brian Conrey who improved this further to two-fifths in 1989 [Wikipedia, 2010b].

[3] Table 3 was obtained from [Borwein et al., 2008] and differs slightly from [Wikipedia, 2010b]

Equivalent Statements

When attempting to prove a statement like the Riemann hypothesis, mathematicians sometimes need to employ a different strategy other than attacking it head on. Instead of proving a statement directly, one way for a mathematician to approach the problem is to find an equivalent statement and prove that. This allows mathematicians, possibly from vastly different specialties, to contribute ideas and help solve the problem. There are many different statements that are equivalent to the Riemann hypothesis. Some equivalent statements are presented here to illustrate how the Riemann hypothesis spans a variety of mathematic disciplines.

One number-theoretic statement that is equivalent to the Riemann hypothesis is, "the number of integers with an even number of prime factors is the same as the number of integers with an odd number of prime factors." This statement can be made precise with the Liouville function, which gives the parity of the number of prime factors of a positive integer. The Liouville function is defined by

$$\lambda(n) = (-1)^{\omega(n)},$$

where $\omega(n)$ is the number of prime factors of n, counted with multiplicity. The Riemann hypothesis is equivalent to the statement that for every fixed $\varepsilon > 0$,

$$\lim_{n\to\infty} \frac{\lambda(1) + \lambda(2) + \cdots + \lambda(n)}{n^{\frac{1}{2}+\varepsilon}} = 0.$$

Another way to say this is that the Riemann hypothesis is equivalent to the statement that an integer has an equal probability of having an odd number or an even number of distinct prime factors [Borwein et al., 2008].

The Riemann hypothesis is very analytic in nature, so it should come as no surprise that there are numerous analytic equivalencies. Hardy and Littlewood provided one analytic equivalence to the Riemann hypothesis that sums the values of $\zeta(s)$, evaluated at odd integers. They showed that the Riemann hypothesis holds if and only if

$$\sum_{k=1}^{\infty} \frac{(-x)^k}{k!\zeta(2k+1)} = O(x^{-\frac{1}{4}}),$$

as $x \to \infty$ [Borwein et al., 2008].

So far, the equivalences have been restricted to the two fields of mathematics, number theory and analysis, for which the Riemann hypothesis provides a connection. However, the Riemann hypothesis can be related to other diverse areas of mathematics that don't seem to have much to do with the zeros of a complex-valued function. First, we define the Redheffer matrix of order n, $R_n = [R_n(i,j)]$, by

$$R_n(i,j) = \begin{cases} 1 & \text{if } j = 1 \text{ or if } i \mid j \\ 0 & \text{otherwise.} \end{cases}$$

Refheffer showed that the Riemann hypothesis is true if and only if

$$\det(R_n) = O(n^{\frac{1}{2}+\varepsilon}),$$

for any $\varepsilon > 0$, which provides a connection with matrix theory. Similarly, Redheffer matrices can be used to relate the Riemann hypothesis to graph theory. Let $B_n = R_n = I_n$ (where I_n is the $n \times n$ identity matrix). Now, let G_n be the directed graph whose adjacency matrix is B_n. Finally, let the graph $\overline{G}_n$ be the graph by adding a loop at node 1 of G_n. The Riemann hypothesis is equivalent to the statement that

$$|\#\left\{\text{odd cycles in } \overline{G}_n\right\} - \#\left\{\text{even cycles in } \overline{G}_n\right\}| = O(n^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$ [Borwein et al., 2008].

There are many more statements equivalent to the Riemann hypothesis than the four that are mentioned above. More equivalencies being developed will only provide mathematicians with alternative strategies for attacking the Riemann hypothesis. Hopefully, this will bring in new ideas from a variety of mathematicians and areas of mathematics and help to solve this 150-year-old puzzle.

Hundreds of Proofs in One

Proving the Riemann hypothesis would be a spectacular achievement in its own right, but the person who proves it would be proving so much more. There are hundreds of papers in the mathematics literature that start with "assume the Riemann hypothesis" and then go on to prove some result. Whomever proves the Riemann hypothesis will effectively be proving hundreds of theorems at once. Here are a few statements that are implied by the Riemann hypothesis.

In 1742, Goldbach conjectured that every natural number $n \geqslant 5$ can be written as the sum of three prime numbers. Euler reformulated this saying that every even number greater than 3 is the sum of two primes. This statement is known as Goldbach's strong conjecture and is one of the oldest unsolved problems in number theory. A weaker version of this statement is that every odd number greater than 7 is the sum of three odd primes. Hardy and Littlewood proved that the generalized Riemann hypothesis implies Goldbach's conjecture for sufficiently large n. In 1997, Deshouillers, Effinger, te Riele and Zinoviev proved that the generalized Riemann hypothesis implies Goldbach's strong conjecture [Borwein et al., 2008].

The performance of many algorithms for primality testing depends on the generalized Riemann hypothesis. The Miller-Rabin primality test is a probabilistic algorithm that runs in deterministic polynomial time if you assume the generalized Riemann hypothesis. Similarly, the probabilistic Solovay-Strassen algorithm is provably deterministic assuming the generalized Riemann hypothesis [Borwein et al., 2008].

Another consequence of the Riemann hypothesis is a better bound for the Dirichlet L-series. Let $L(1, \chi_D)$ be the value of the Dirichlet L-series

$$L(1, \chi_D) = \sum_{n=1}^{\infty} \frac{\chi_D(n)}{n},$$

at 1, where χ_D is a non-principal Dirichlet character with modulus D. It is known that $|L(1, \chi_D)|$ is bounded by

$$D^{-\varepsilon} \ll_{\varepsilon} |L(1, \chi_D)| \ll logD$$

However, assuming the Riemann hypothesis, Littlewood proved that the lower bound on $|L(1, \chi_D)|$ can be improved to be

$$\frac{1}{\log \log D} \ll |L(1, \chi_D)| \ll \log \log D.$$

There are hundreds of other statements and conjectures that mathematicians have proven by assuming the Riemann hypothesis. If it is ever proved, then a vast number theorems will automatically be added to the canon of mathematics. While mathematicians don't need any additional incentive to work on this problem, having such a large body of work depend on it will only generate more interest and make it that much more important.

It is not clear if a proof of the Riemann hypothesis will ever be found. However, the evidence seems to suggest that it is true. The Riemann hypothesis has deep ties to the distribution of prime numbers and is one of the most important unsolved problems today. It was considered important enough to be included on Hilbert's list of 23 unsolved problems that he presented at the International Congress of Mathematicians in Paris in 1900 and was also chosen by the Clay Mathematics Institute in 2000 to be one of the seven Millennium Prize Problems, each worth $1,000,000 US. It has inspired mathematicians for over a century and even inspired a song (included below) by the American analytic number theorist Tom Apostol [Robbin, 2010]. The Riemann hypothesis will certainly be the focus of mathematical research for many years to come.

Where are the zeros of zeta of s?

(to the tune of *Sweet Betsy from Pike*)

Where are the zeros of zeta of s?
G.F.B. Riemann has made a good guess,
They're all on the critical line, sai he,
And their density's one over 2pi log t.

This statement of Riemann's has been like trigger
And many good men, with vim and with vigor,
Have attempted to find, with mathematical rigor,
What happens to zeta as mod t gets bigger.

The efforts of Landau and Bohr and Cramer,
And Littlewood, Hardy and Titchmarsh are there,
In spite of their efforts and skill and finesse,
(In) locating the zeros there's been no success.

In 1914 G.H. Hardy did find,
An infinite number that lay on the line,
His theorem however won't rule out the case,
There might be a zero at some other place.

Let P be the function pi minus li,
The order of P is not known for x high,
If square root of x times log x we could show,
Then Riemann's conjecture would surely be so.

Related to this is another enigma,
Concerning the Lindelof function mu(sigma)
Which measures the growth in the critical strip,
On the number of zeros it gives us a grip.

But nobody knows how this function behaves,
Convexity tells us it can have no waves,
Lindelof said that the shape of its graph,
Is constant when sigma is more than one-half.

Oh, where are the zeros of zeta of s?
We must know exactly, we cannot just guess,
In orer to strengthen the prime number theorem,
The integral's contour must not get too near 'em.

Tom Apostol

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