ABELIAN VARITIES AND COMPLEX TORII

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1. Introduction

The Elliptic curves over $\mathbb{C}$ are isomorphic to the basic complex torus (that is, the torus can be embedded into the complex projective plane. Abelian varieties are the generalization of Elliptic curves to higher dimension, so it is natural to ask whether the d-dimensional Abelian varieties are isomorphic to the d-dimensional complex torii, or if not, what's the closest condition that holds.

By wandering through the application of a series of tools and constructions, particularly Riemann forms, divisors, and theta functions, it is possible to classify the subset of the d-dimensional complex torii that correspond to d-dimensional Abelian varieties. Specifically, it is both necessary and sufficient that the torus have a positive definite Riemann form defined on it; the set of such torii corresponds exactly to the set of Abelian varieties.

2. Tools

First, some labels. Let V be a d-dimensional vector space over $\mathbb{C}$. Let Λ be a lattice in V. Let T be the complex torus V/Λ.

A Riemann form on T is defined to be a Hermetian form H on V (with $H(u,v) = S(u,v) + iE(u,v)$ split into its real and imaginary parts), such that $E(\lambda_1, \lambda_2) \in \mathbb{Z}$ for all λ_1, λ_2 in Λ. Additionally, if H is positive definite, then the Riemann form is called non-degenerate (or positive definite). (We will call H the Riemann form in addition to being the Hermetian form.) T is called an Abelian manifold if it posesses a positive definite Riemann form.

(A refresher: To be Hermetian, $H(u,v)$ is linear on u, antilinear on v, and $H(u,v) = \overline{H(v,u)}$. Additionally, the component functions obey $S(u,v) = E(iu,v)$, $S(iu,iv) = S(u,v)$, $E(iu,iv) = E(u,v)$, S is symmetric, and E is antisymmetric. Lastly, H is positive definite if $H(u,u) > 0$ if $u \neq 0$.)

Now, divisors. Let π be the projection map from V to $T = V/\Lambda$. A function f is periodic on T, as expected, if it is periodic with respect to Λ ($f(z+\lambda) = f(z)$ for all $\lambda \in \Lambda$). Functions on T can be converted to periodic functions on V by composition with π . A divisor D on V is an open covering $\cup U_\alpha$ of V along with a set of functions f_α such that f_α is meromorphic on U_α and on $U_\alpha \cap U_\beta$, f_α / f_β is holomorphic (that is, f_α and f_β have matching zeros and poles on the intersection). A divisor is periodic

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if it is unchanged under translations by elements of Λ (shifting the cover and the functions in the obvious way). The divisor of a periodic meromorphic function is a periodic divisor.

Cousin's Theorem (see [2]) states that on V, all divisors are somewhat simpler that the general case in that all of the f_α are the same- a single f that is meromorphic on V. This can be used to show that on any periodic divisor D on V, $f(z+\lambda) = U_\lambda(z)f(z)$ with $U_\lambda(z) = e^{2\pi i h_\lambda(z)}$ and $h_\lambda(z)$ is holomorphic, and $h_{\lambda_1+\lambda_2}(z) = h_{\lambda_1}(z+\lambda_2) + h_{\lambda_2}(z) \mod \mathbb{Z}$. For our purposes, we will consider h_λ of the form $h_\lambda(z) = L(z,\lambda) + J(\lambda)$ with L z-linear and J z-constant.

Finally, theta functions. Given an L and J as before, a theta function of type (L, J) is a function θ (surprise) on V with $\theta(z+\lambda) = e^{2\pi i (L(z,\lambda)+J(\lambda))}\theta(z)$ for all $z \in V$, $\lambda \in \Lambda$. It must be either meromorphic or holomorphic.

3. Results

So what do we do with all this stuff? Chain it all together, mostly. Proofs are going to be omitted mostly due to laziness. Sketches of all can be found in [2] with more thorough treatments of most in [1] and [3].

The first is from Poincaré. Every periodic divisor D on V extended from a divisor D' on T is the divisor of a meromorphic theta function θ. If D' is holomorphic, then so is θ.

Now, if θ is a theta function of type (L,J), we can associate a Riemann form to it in the following way. Define $E(z,w) = L(z,w) - L(w,z)$. E works out to be anti-symmetric, $\mathbb{R}$-bilinear, and assumes integer values on $\Lambda \times \Lambda$. Lastly, $E(iz, iw) = E(z,w)$. All of this is as desired. If H is defined as $H(z,w) = E(iz,w) + iE(z,w)$, then H is a (Hermetian, hence) Riemann form on T.

Additionally, if D is a holomorphic divisor, then the corresponding Riemann form is positive.

If the Riemann form is positive definite, then the vector space of all holomorphic theta functions has dimension $\mathrm{Pf}(E)$, where Pf is the Pfaffian operator, which I don't claim to understand at this point, heh (into the hand-waving weeds). This theorem is due to Frobenius.

Ultimately, these connections can be tied together in this way: if D is a divisor associated to a positive definite Riemann form, then there is a holomorphic theta function associated. The space of all holomorphic theta functions of the same type (type (L,J)) is of sufficiently large degree that it can be used to construct a projective embedding of the torus T that the Riemann form came from. This is equivalent to saying that the torus is isomorphic to an Abelian variety, as desired.

Super-handwavy, but there it is.

The other direction (All Abelian varieties are isomorphic to torii) is much more direct, coming from exponentiation being a map from V to the Abelian variety, with a discrete subgroup (lattice) as its kernel.

So, ultimately, if you can attach a Riemann form to the d-dimensional complex vector space V with lattice Λ, and that Riemann form is positive definite, then (and only then) is the torus V/Λ isomorphic to an Abelian variety.

References

[1] D. Mumford. Abelian Varieties, Oxford University Press, London, 1970.
[2] M. Rosen. Abelian Varieties over C. in *Arithmetic Geometry, ch 4, J H Silverman, Editor*, Springer-Verlag, 1998.
[3] H. P. F. Swinnerton-Dyer. Analytic Theory of Abelian Varieties. Cambridge University Press, 1974.