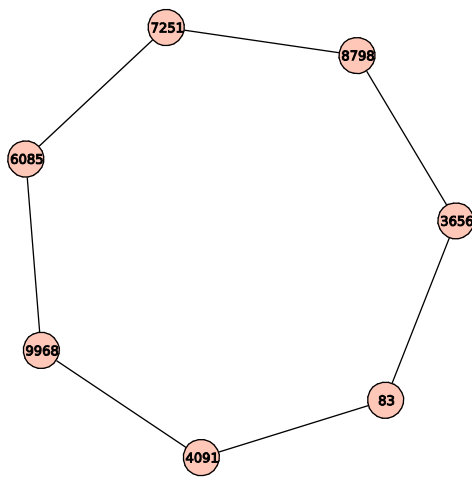


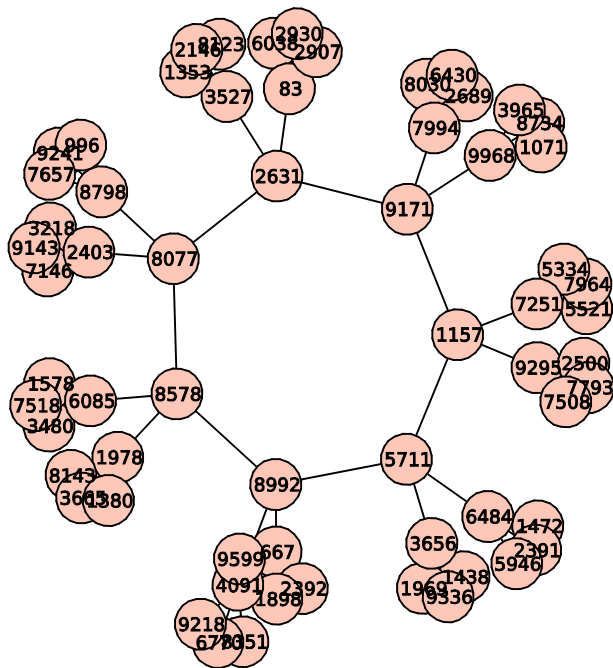
This is what a typical isogeny graph for an ordinary curve looks like. It is referred to as a ‘volcano’. You can visualize the volcano by noticing that it has a crater, and the rest of the graph can be seen to build the sides of the volcano. The curves in the crater all have the same endomorphism ring—namely the full ring of integers. Then each time that you descend an edge in going down the side of the volcano, the endomorphism ring is changed into an order with conductor l inside the maximal order.

Here’s another example. Same curve, $l=7$

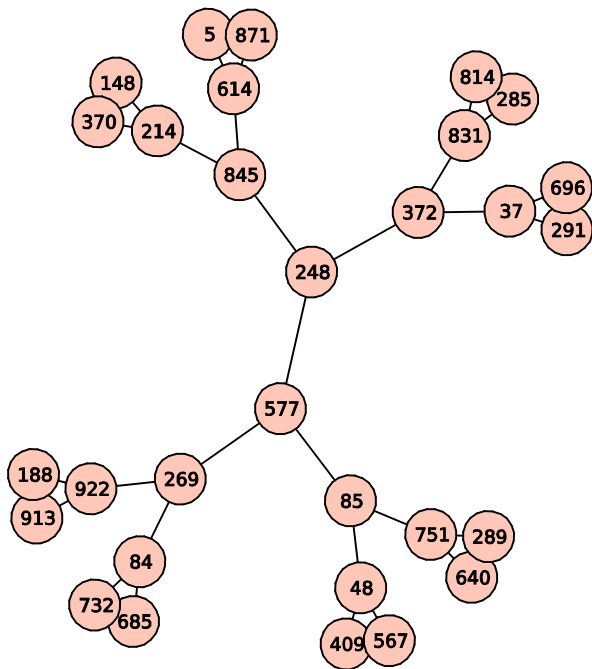


This time, there is only a crater. So all the curves here have endomorphism ring equal to the ring of integers. We can determine the quadratic field once we know the trace t of the Frobenius. Then the field is $\mathbb{Q}(\sqrt{k})$, where $k = t^2 - 4$.

Another example of a volcano. `EllipticCurve(GF(1009),[363,690])` $l=3$



Here's an example where the crater is just a two-cycle. $\text{EllipticCurve}(\text{GF}(1009), [1, 3]) \text{ } l=2$



Note that except for the vertices on the ground level of the volcano, all vertices have $l+1$ edges. This is true of the previous example. This is best explain by considering modular polynomials. If j is the j -invariant of your curve E , then the l -th modular polynomial evaluated at j has $l+1$ roots, which are the j -invariants of curves l -isogenous to E .

The function `number_ss_curves()` returns the number of supersingular curves over \mathbb{F}_p . For supersingular curves, all the curves are actually defined over \mathbb{F}_p . This number is approximately equal to $p/12$.

```
sage:number_ss_curves(1009)
```

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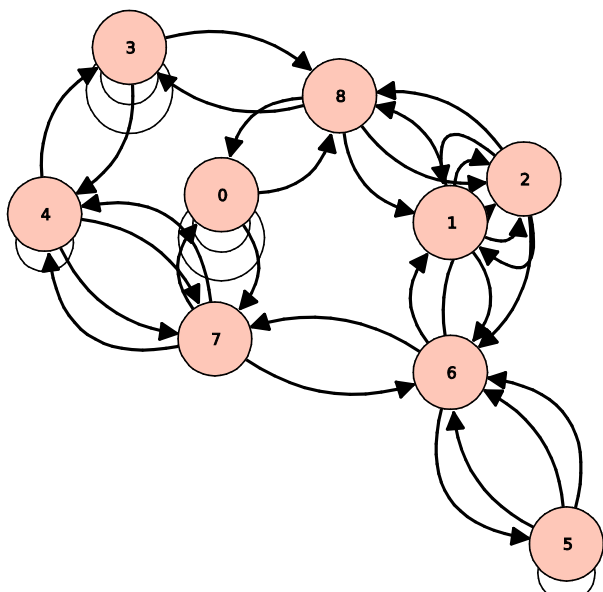
The function `find_ss_curve(p)` returns a supersingular elliptic curve over \mathbb{F}_p . A more general algorithm to find a supersingular curve with any possible (allowable) trace can be found in a paper by R. Brooker, and used in my dissertation. I have code for it as well.

```
sage:find_ss_curve(1009)
```

returns `EllipticCurve(GF(1009),[514,495])`, which is supersingular

We can also look at the isogeny graph for supersingular curves. For supersingular curves, the graph of l -isogenies is connected. This algorithm could be created via modular polynomials or by using Brandt matrices.

```
sage:ss_graph(101,3)
```



The function `hurwitz_class_number()` returns the total number of curves in an isogeny class. If the curve E has trace t , and is defined over F_p , then the input to the algorithm is t^2-4p .

```
sage: hurwitz_class_number(-2700)
```

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This calculation is verified by the next example. It is a program which returns a list of all the j -invariants of curves l -isogenous to E . The program uses various values of l and isogeny graphs to find them all

```
sage: E=EllipticCurve(GF(691),[2,114])
```

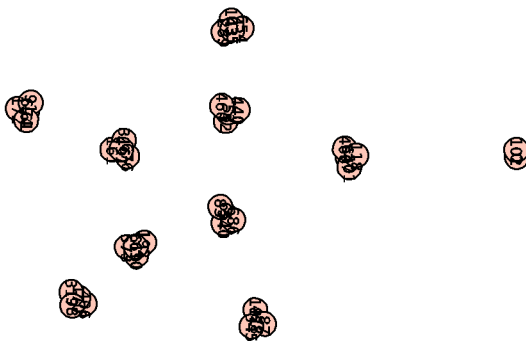
```
sage: list_of_isogenous_curves(E)
```

```
[0, 647, 172, 651, 654, 143, 406, 540, 161, 674, 676, 170, 428, 686, 573, 52, 53, 440, 316, 317, 181, 192, 289, 460, 98, 83, 468, 87, 345, 91, 610, 101, 102, 619, 61, 497, 118, 635]
```

If you count them, there are indeed 38.

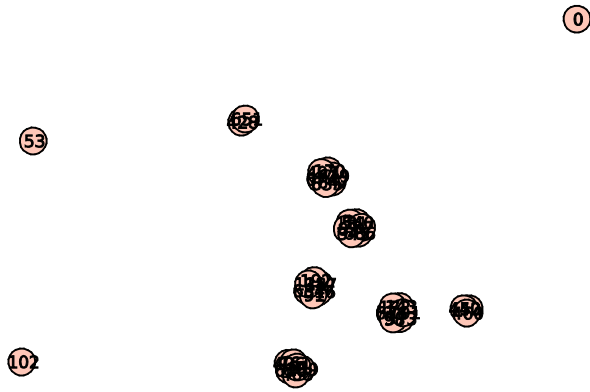
Now above, when we were constructing the isogeny graph of an elliptic curve, we were just finding a connected component in a larger graph. The vertices connected to E are the other curves which are l -isogenous to E . However, there are other curves defined over F_p which have the same trace as E , and they have their own connected components. The total graph is called a cordillera (which is spanish for mountain range). It is a collection of volcanoes. Here's an example where E is the same curve as in our last example. It finds all the 2-isogeny volcanoes.

```
sage: cordillera(E,2)
```



The graph is hard to see, but we count that there are 10 volcanoes. Now, same curve, different l .

`cordillera(E,7)`



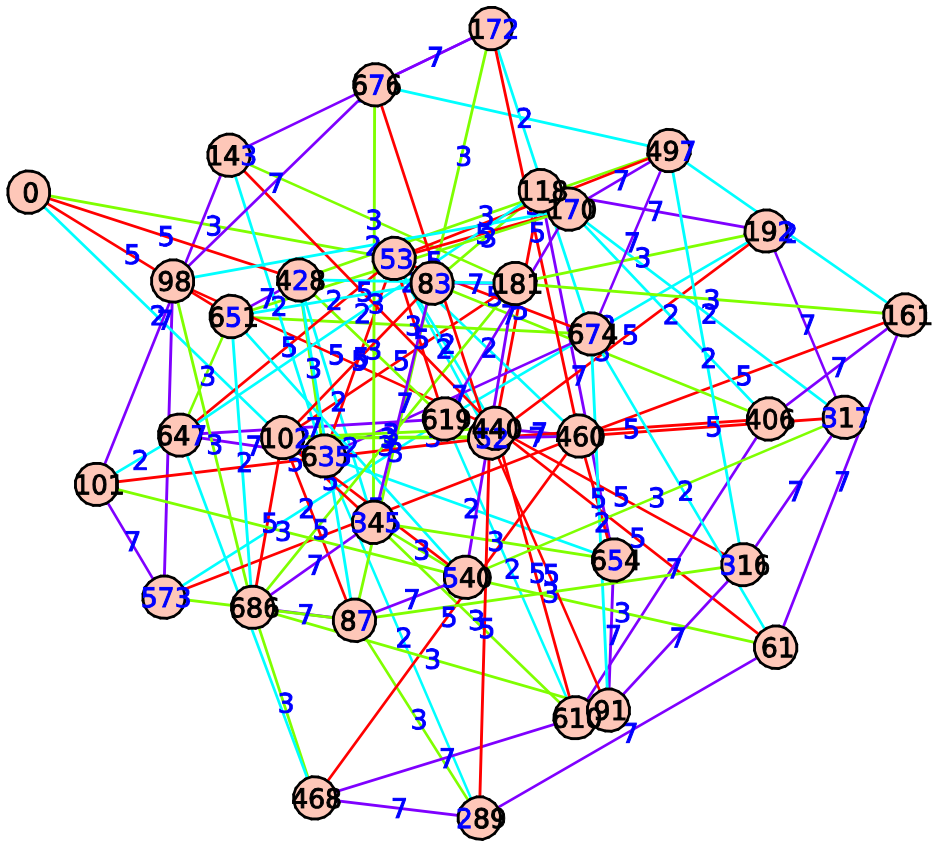
We count that there are 10 volcanoes of l -isogenies. I wasn't satisfied with the algorithm above, so I created another algorithm to compute cordilleras more efficiently. This is the function `cordillera_alt()`

`Z=list_of_isogenous_curves(E)`

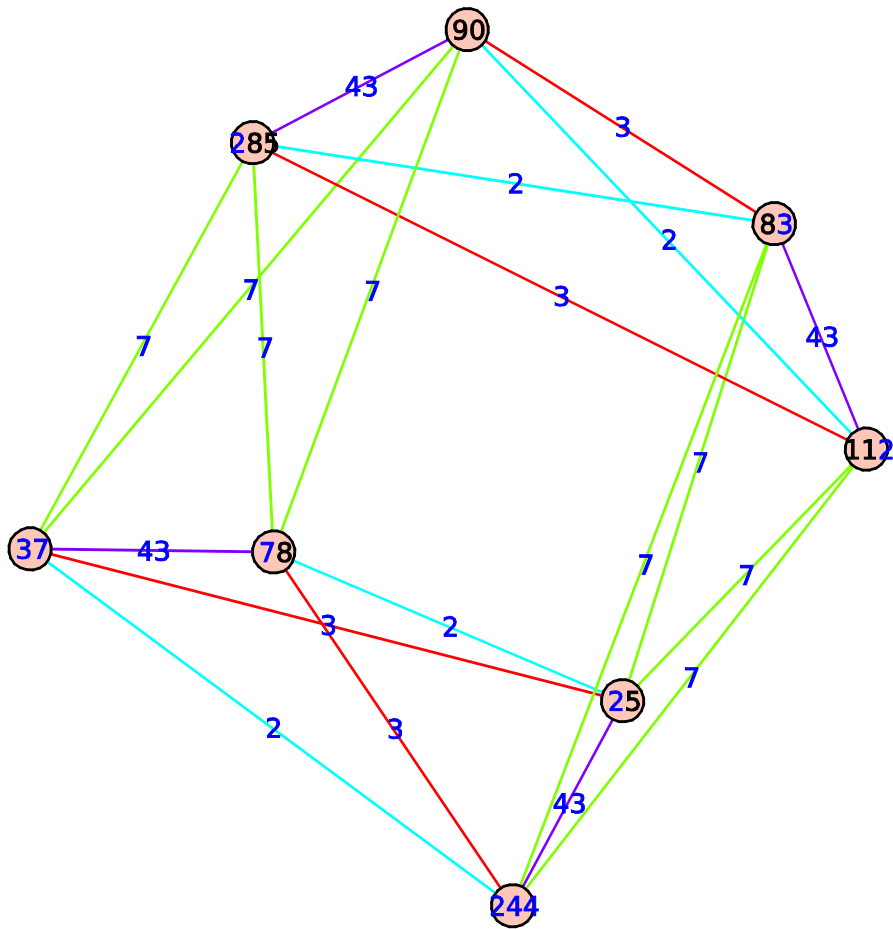
`G,Z=cordillera_alt(691,Z,2)`-produces same graph as the $l=2$ one above

As a final program, we can put all the l -cordilleras together on the same vertices. This gives us some nice colorful pictures. If I could manipulate the graph better, I'm sure there would be much more symmetry. Inside, if you fix l , you find several volcanoes. Examples are given below:

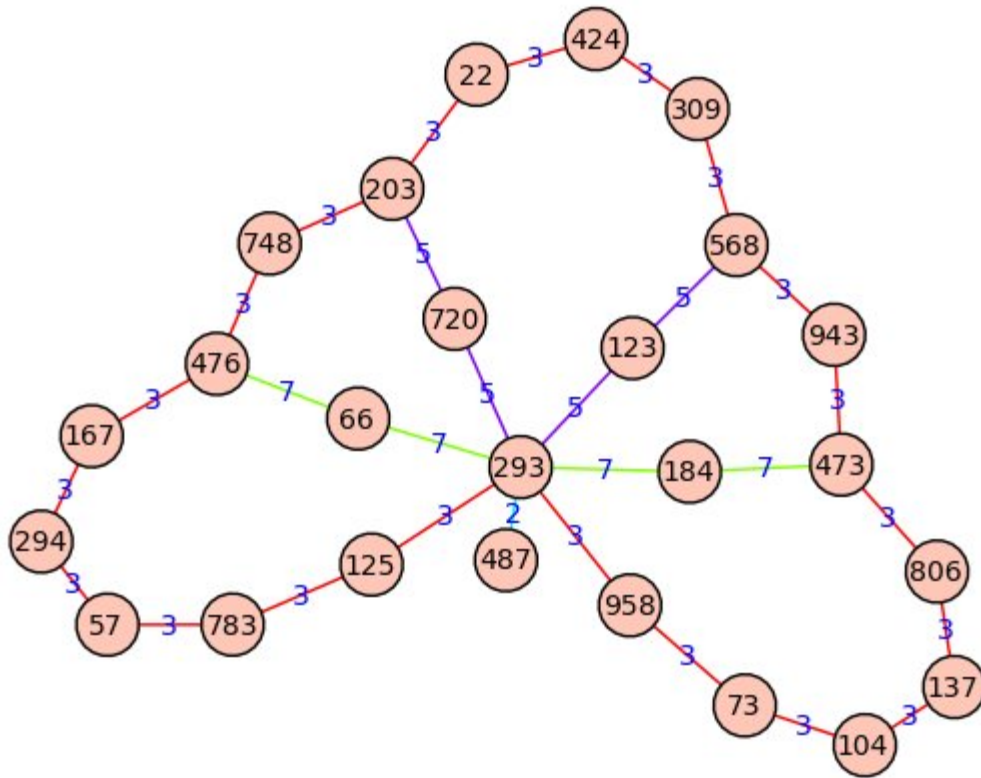
`complete_isogeny_graph(E)`



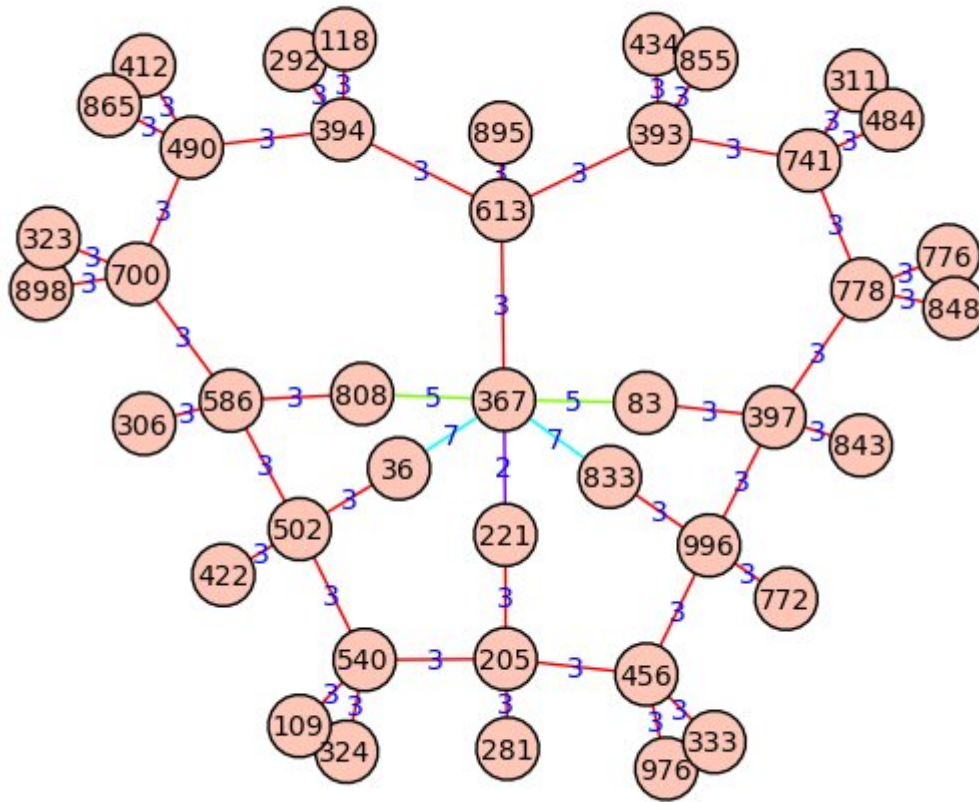
EllipticCurve(GF(307),[1,5])



E=EllipticCurve(K,[2,115])



$E = \text{EllipticCurve}(K, [1125, 1214])$ $p = 1009$



Even these pictures don't show everything. If I used curves defined over fields of bigger characteristic, the graphs get too cluttered to see anything. Note that all the curves in this graph have the same cardinality because two isogenous curves have the same cardinality.