Adèles and the Finiteness of the Class Number

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Abstract

This note is trying to be slick, so all the proofs are most efficient and neat.

1 Adèles

Let K be a number field, i.e., an extension $K/\mathbb{Q}$ of degree n.

For each prime ideal $\mathfrak{p}$ of $\mathcal{O}_K$, the ring of integers of K, we have a valuation $v_\mathfrak{p} : K \to \mathbb{Z}$ given by $v_\mathfrak{p}(x)$ is the exponent of $\mathfrak{p}$ in the prime decomposition of the ideal $x\mathcal{O}_K$. Each $v_\mathfrak{p}$ has an associated metric $|x|'_\mathfrak{p} = 2^{-v_\mathfrak{p}(x)}$. The field K is not complete with respect to the metric $v_\mathfrak{p}$ so we can take $K_{v_\mathfrak{p}}$ to be the completion of K.

There exist embeddings $i_1, \ldots, i_{r_1} : K \hookrightarrow \mathbb{R}$ and $j_1, \ldots, j_{r_2} : K \hookrightarrow \mathbb{C}$ (then $r_1 + 2r_2 = n = [K : \mathbb{Q}]$) and natural inclusions $i_\mathfrak{p} : K \hookrightarrow K_{v_\mathfrak{p}}$. We will call each $i_k, j_k, v_\mathfrak{p}$ a place and we will denote a general place by v. If v is of the form i_k we call it a real place and if it is of the form j_k we will call it a complex place. We will call these infinite places; each place v of the form $v_\mathfrak{p}$ is called a finite place and the valuation $v_\mathfrak{p}$ will be denoted simply by v; it has the property that $v(xy) = v(x) + v(y)$ and $v(x + y) \geq \min(v(x), v(y))$.

Every finite place has, by definition, an associated prime ideal $\mathfrak{p}_v$. We will write K_v for the completion of K. Consider the ring of integers $\mathcal{O}_v = \{x \in K_v | v(x) \geq 0\}$ which is a local ring with maximal ideal $\wp_v = \{x \in K_v | v(x) > 0\}$. It is a principal ideal domain and a generator π_v of $\wp_v$ is called a uniformizer of K_v. Every fractional ideal of $\mathcal{O}_v$ is generated by π_v^m for some $m \in \mathbb{Z}$. Let $k_v = \mathcal{O}_v/\wp_v$ be the (finite) residue field at v and let $q_v = \#k_v$.

For each real place v we write $K_v = \mathbb{R}$ and for each complex place we write $K_v = \mathbb{C}$. Each field K_v has a canonical norm on it. For v real it is $|x|_v = |i_v(x)|$ where i_v is the real embedding. For v complex it is $|x|_v = |i_v(x)|^2$, where i_v is the complex embedding. For v finite it is $|x|_v = q_v^{-v(x)}$.

Let S be a finite set of places that includes all the infinite places. Define

$$\mathbb{A}_{K,S} = \prod_{v \in S} K_v \prod_{v \notin S} \mathcal{O}_v.$$

Endow $\mathbb{A}_{K,S}$ with the product topology.

Define the ring of adèles over K to be

$$\mathbb{A}_K = \bigcup_S \mathbb{A}_{K,S},$$

together with the topology consisting of sets U such that $U \cap \mathbb{A}_{K,S}$ is open in $\mathbb{A}_{K,S}$ for all finite sets S that contains the infinite places.

Lemma 1.1. *1. The ring $\mathbb{A}_K$ is Hausdorff.*

2. The ring $\mathbb{A}_K$ is locally compact.

Proof. 1. Points in $\mathbb{A}_K$ are sequences (x_v) such that $x_v \in \mathcal{O}_v$ for almost all v. Let $(x_v) \neq (y_v)$ be two such points and assume that $x_\mu \neq y_\mu$ for some finite place μ. Then consider $U_\mu \ni x_\mu$ and $V_\mu \ni y_\mu$ be disjoint neighborhoods (The rings $\mathcal{O}_v$ are metric spaces and so Hausdorff). Then the preimages of U_μ and V_μ under the projection map to the μ component will separate (x_v) and (y_v).

2. Around each point (x_v) the neighborhood $\prod_{v=\infty}\{x_v\}\prod_{v<\infty}\mathcal{O}_v$ is compact by Tychonov's theorem.

□

An annoying to prove, but true, fact is that $\mathbb{A}_K$ is a topological group under component-wise addition and multiplication. Define $\mathbb{A}_K^\times$ to be the multiplicative subgroup of $\mathbb{A}_K$, consisting of all sequences (x_v) such that $v(x_v) = 0$ for almost all finite v. The topology on $\mathbb{A}_K^\times$ is the direct limit product topology on the multiplicative groups $K_v^\times$ and $\mathcal{O}_v^\times$.

For every $x \in K$ there are only finitely many prime ideals that divide $x\mathcal{O}_K$ so $K \hookrightarrow \mathbb{A}_K$ but also $K^\times \hookrightarrow \mathbb{A}_K^\times$.

2 Topology

We would like to understand the topological properties of $K \subset \mathbb{A}_K$ and $K^\times \subset \mathbb{A}_K^\times$.

Proposition 2.1. *K is discrete in $\mathbb{A}_K$ and $\mathbb{A}_K/K$ is compact.*

Proof. We will interpret K and $\mathcal{O}_K$ as embedded in $\mathbb{A}_K$. Let $\mathbb{A}_\infty = \prod_{v=\infty} K_v$. By the Chinese Remainder Theorem we essentially get that $\mathbb{A}_K = K + \mathbb{A}_{K,\emptyset}$. Clearly $K \cap \mathbb{A}_{K,\emptyset} = \mathcal{O}_K$. But $\mathcal{O}_K$ is discrete in $\mathbb{A}_\infty$ so it is discrete in $\mathbb{A}_{K,\emptyset}$ so K is discrete in $\mathbb{A}_K$.

Let $C = \prod_{v=\mathbb{R}}[-1/2, 1/2]\prod_{v=\mathbb{C}}\{|z| \leq 1/2\}\prod_{v<\infty}\mathcal{O}_v$ which is compact by Tychonov. Then note that $\mathbb{A}_{K,\emptyset} = \mathcal{O}_K + C$, again by the Chinese Remainder Theorem and so $\mathbb{A}_K = K + C$. This means that $\mathbb{A}_K/K = (K+C)/K$ is compact being a closed subset of a compact set. □

Locally compact groups, such as $\mathbb{A}_K$ have something called a Haar measure, which is a $d\mu(x)$, which is unique up to multiplication. As such, for every $y \in \mathbb{A}_K$ if we look at $d\mu(yx)$ we get another measure, so by uniqueness there exists a scalar $|y| \in \mathbb{R}_{>0}$ such that $d\mu(yx) = |y|d\mu(x)$. Then we have the property that

$$|x| = \prod_v |x_v|_v,$$

where $x = (x_v)$. Note that $|x|$ is convergent since almost all terms in the product at ≤ 1.

Lemma 2.2. *Let $\mathbb{A}_K^1 \subset \mathbb{A}_K^\times$ be the set $\{x \in \mathbb{A}_K^\times | |x| = 1\}$ whose topology is inherited from $\mathbb{A}_K$ and $\mathbb{A}_K^\times$ (the inherited topologies are the same). Then $K^\times \hookrightarrow \mathbb{A}_K^1$.*

Proof. Conceptually this is a simple problem. But we will use the neatest method.

Since $\mathbb{A}_K/K$ is compact, it will have a finite volume relative to $d\mu(x)$. But for every $y \in K$ we have $\alpha_y : x \mapsto yx$ is an automorphism of $\mathbb{A}_K/K$. (If $x \in K$ then $xy \in K$ and vice-versa so it is well-defined.) Therefore,

$$\int_{\mathbb{A}_K/K} d\mu(x) = \int_{\mathbb{A}_K/K} d\mu(\alpha_y(x)),$$

since all we are doing is a change of variables. But then $\int_{\mathbb{A}_K/K} d\mu(x) = |y| \int_{\mathbb{A}_K/K} d\mu(x)$ so $|y| = 1$. □

We also have a discrete embedding $K^\times \hookrightarrow \mathbb{A}_K^1$ by the above Lemma.

Lemma 2.3. *Let $a \in \mathbb{A}_K$ such that*

$$|a| > \frac{\text{vol}(\mathbb{A}_K/K)}{\text{vol}(C)}.$$

Prove that there exists an $x_a \in K$ such that $|x_a|_v \leq |a_v|_v$.

Proof. Let $A_a = aC$. Then $\text{vol}(A_a) > \text{vol}(A_K/K)$. Therefore the map $aA \to \mathbb{A}_K/K$ is not an injection so there exist $u, v \in A_a$ such that $u - v \in K$. But then by construction of C we have

$$|(u - v)|_v \leq |a_v|_v,$$

for every place v so $u - v \in aA \cap K$. □

Proposition 2.4. *$\mathbb{A}_K^1/K^\times$ is a compact topological group.*

Proof. This is no longer as simple as $\mathbb{A}_K/K$ compact, but it is of a similar flavor. Instead of taking the surjection $K + C \to \mathbb{A}_K/K$ to imply that $\mathbb{A}_K/K$ is compact, we will look for a compact set W and a surjection $W \to \mathbb{A}_K^1/K^\times$.

Take $W = \{x \in \mathbb{A}_K^\times | |x_v|_v \leq |a_v|_v\}$ where $a \in \mathbb{A}_K^\times$ such that

$$|a| > \frac{\text{vol}(\mathbb{A}_K/K)}{\text{vol}(C)}.$$

By the previous Lemma the map $W \to \mathbb{A}_K^1/K^\times$ must be surjective because if $t \in \mathbb{A}_K^1$ then there exists x_a such that $|x_a|_v \leq |a_v/t_v|_v$ so $x_a t \in W$. □

3 The Finiteness of the Class Number of K

Let $Cl(K)$ be the class group of K and let $I(K)$ be the group of fractional ideals of K.

Consider the map $\text{id} : \mathbb{A}_K^\times \to I(K)$ defined by

$$\text{id} : (x_v) \mapsto \prod_{v<\infty} \mathfrak{p}_v^{v(x_v)}.$$

This map is well-defined because the product of prime ideals is a finite one. Moreover, if we restrict to $\text{id} : \mathbb{A}_K^1 \to I(K)$ we get a surjection because the preimage of $\prod \mathfrak{p}_v^{e_v}$ can be taken to be $a \in \mathbb{A}_K$ such that $a_v = 1$ for all infinite places $v \neq i_1$, $a_v = q_v^{e_v}$ for all finite places v and $a_{i_1} \in \mathbb{R}$ to be whatever is needed to make $a \in \mathbb{A}_K^1$.

The kernel of this map is clearly $A_\infty \prod_{v<\infty} \mathcal{O}_v^\times$ so we get a bijection

$$\text{id} : \mathbb{A}_K^1/(A_\infty \prod_{v<\infty} \mathcal{O}_v^\times) \to I(K).$$

We get a projection $\text{id} : \mathbb{A}_K^1/(A_\infty \prod_{v<\infty} \mathcal{O}_v^\times) \to \text{Cl}(K) = I(K)/K^\times$. What is its kernel? We need all $a \in \mathbb{A}_K^1$ such that $a = (a_v)$ maps to a principal ideal.

But principal ideals correspond to a factorization $x\mathcal{O}_K = \prod_{v<\infty} \mathfrak{p}_v^{v(x)}$ and the (unique) preimage via the injective map id in $\mathbb{A}_K^1/(A_\infty \prod_{v<\infty} \mathcal{O}_v^\times)$ is $x \in K^\times$. Therefore we get a bijection

$$(\mathbb{A}_K^1/K^\times)/(A_\infty \prod_{v<\infty} \mathcal{O}_v^\times) \to \text{Cl}(K).$$

Note that $A_\infty \prod_{v<\infty} \mathcal{O}_v^\times$ is open in $\mathbb{A}_K^1/K^\times$ which is compact so $\text{Cl}(K)$ must be finite.